



Polarities in structural analysis and design: n -dimensional graphic statics and structural transformations



Marina Konstantatou^{a,*}, Pierluigi D'Acunto^b, Allan McRobie^a

^a University of Cambridge, Department of Engineering, Trumpington St, Cambridge CB2 1PZ, UK

^b ETH Zürich, Institute of Technology in Architecture, Chair of Structural Design, Stefano-Franscini-Platz 5, Zürich 8093, CH, Switzerland

ARTICLE INFO

Article history:

Received 8 March 2018

Revised 11 June 2018

Available online 12 July 2018

Keywords:

Structural design

Graphic statics

Reciprocal diagrams

Airy stress function

Static equilibrium

Maxwell

Rankine

Cremona

Poncelet duality

Projective geometry

ABSTRACT

This paper proposes a unified and entirely geometrical methodology for generating 2D and 3D force diagrams for given planar and spatial trusses in static equilibrium within the context of graphic statics. The trusses, regarded as form diagrams, are projections of higher dimensional simply-connected stress functions, can be either self-stressed or loaded with external forces, and need not be tension-only/compression-only. First, we provide an in-depth overview of fundamental notions of graphic statics and projective geometry related to the construction of reciprocal form and force diagrams. Specifically, we describe a series of polar transformations and discuss them from a geometric and an algebraic standpoint. Moreover, we provide an exegesis and visualization of Maxwell's reciprocal methods while discussing them within the scientific framework of his time. We then develop a contemporary graphic statics framework grounded on projective geometry and higher-dimensional reciprocal stress functions. Within this framework, we provide a general methodology for deriving any type of reciprocal diagrams for the planar and the spatial case. Advantages of this method include: direct control on any of the four reciprocal objects (form diagram, force diagram, corresponding stress functions) and no need for iterative/procedural reconstruction of any of the diagrams since it is based on direct geometrical transformations. Finally, we highlight how duality, and in particular polarity, can be used for a wide range of applications in structural analysis and design.

© 2018 The Author(s). Published by Elsevier Ltd.

This is an open access article under the CC BY license. (<http://creativecommons.org/licenses/by/4.0/>)

1. Introduction

Graphic statics is a geometrical framework for the design and analysis of 2D and 3D trusses in static equilibrium mostly developed during the 19th century by natural philosophers, scientists, and practitioners of engineering such as (Culmann, 1857; Rankine, 1864; Maxwell, 1864; Maxwell, 1870; Cremona, 1872) among others (Kurrer, 2008; Charlton, 1982). In its initial widespread implementation, graphic statics consisted of a series of graphical procedures for the construction and transformation of two reciprocal diagrams, the form and force diagrams, the former representing the geometric layout of a loaded truss and the latter the equilibrium of the internal and external forces applied to the nodes of the truss. Nowadays, the possibility to support graphic statics with contemporary computational tools has made it possible to develop visual and intuitive approaches for the design, analysis, and optimisation of structures based on the interplay between the

reciprocal diagrams. However, most of the current methodologies are generally case specific, work either for 2D or 3D form diagrams and do not make use of the possibilities offered by the underlying fundamental geometrical framework, firstly introduced by Maxwell (1864) and Maxwell (1870), based on projective geometry and reciprocal stress functions.

1.1. Objectives and contributions

The aim of this paper is to propose a unified and entirely geometrical graphic statics framework for constructing 2D and 3D reciprocal force diagrams for given planar and spatial trusses in static equilibrium. Taking advantage of the methods provided by projective geometry and the notion of higher-dimension stress functions, the proposed unified framework can be used to derive any type of planar or spatial reciprocals of a given truss in equilibrium, either self-stressed or externally loaded, when it is a projection of a simply-connected higher dimensional stress function. This methodology applies to numerous structures, however it should be noted that there are also some configurations of trusses in equilibrium which cannot be represented by reciprocal diagrams. Since this

* Corresponding author.

E-mail address: mk822@cam.ac.uk (M. Konstantatou).

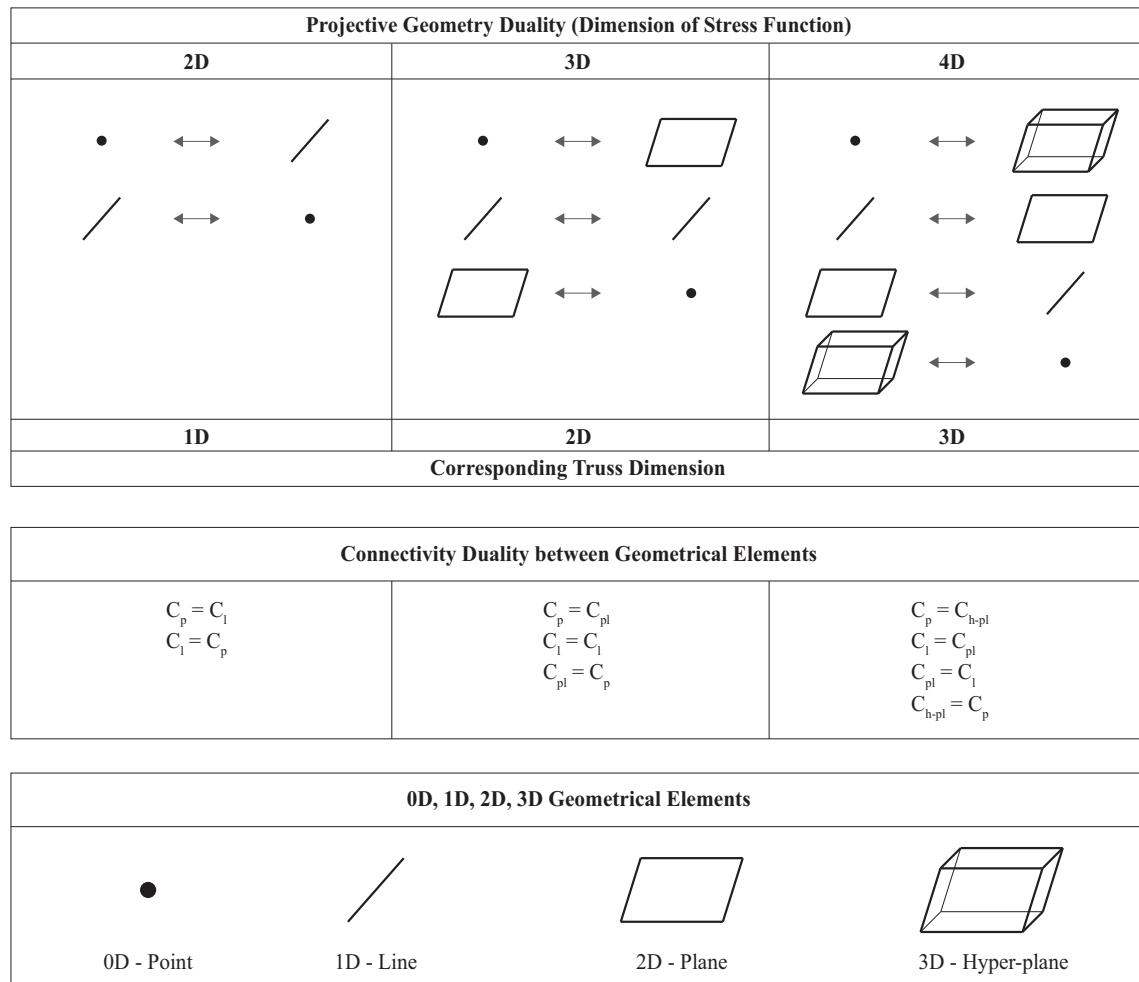


Fig. 1. Summary of duality principles between geometrical elements in projective geometry and correspondence between n-dimensional structures and their (n+1)-dimensional stress functions.

method is based on direct geometrical transformations there is no need for iterative or procedural reconstruction of the reciprocal diagrams every time one of them is updated. On the contrary, the designer can have direct control on any of the reciprocal objects: form diagram; force diagram; corresponding stress functions and any local or global transformation in one of them will directly update the others. Apart from design, analysis, and optimization of structures, the suggested framework also encompasses other applications in structural engineering theory, such as the generation of dual structures based on the notion of polar transformations.

1.2. Content

To set up a consistent theoretical ground for the proposed framework, in Section 2 we summarise a series of fundamental notions related to graphic statics, reciprocal diagrams and Airy stress functions, as well as projective geometry and polar transformations. These notions are contextualized within the scientific framework of the time when they were introduced and in particular in relation to Maxwell’s approach. From this starting point, we propose in Section 3 a unified methodology for deriving any type of planar or spatial reciprocals of trusses, which are projections of higher-dimensional stress functions. In Section 4, we discuss how this framework can be used for global and local transformations of any one of the reciprocal objects and how it can potentially simplify and give insights into the design, analysis, and optimisation of structures by considering specific case studies. Finally, in

Section 5 we highlight existing applications of polar transformations for the creation of dual structures and how these can be directly included in the proposed framework.

2. Fundamental notions

2.1. Graphic statics

The scientific context that gave rise to the development of graphic statics is grounded on the legacy of Da Vinci, Galilei, Newton (Zalewski and Allen, 1998), Hooke, Poleni, and Stevin (Heyman, 1995). Cremona (1872) traces back the foundation of graphic statics to the 18th century, when Varignon used the funicular polygon and the polygon of forces to calculate and visualize the equilibrium of a system of forces in plane (Varignon, 1725). According to Maxwell and Cremona, Rankine was the first one to propose a consistent graphical methodology for the analysis of bar frameworks in his “A Manual of Applied Mechanics” (Rankine, 1858). The earliest systematisation of the theory of graphic statics was due to Culmann, with the publication of his seminal monograph ‘Die Graphische Statik’ (Culmann, 1866). Rankine’s, Maxwell’s, Cremona’s, and Culmann’s methods were then further extended by various scientists and practitioners of engineering at the end of the 19th century, such as Bow in his work ‘The economics of construction in relation to framed structures’ (Bow, 1873). However, it is Maxwell who is credited as the originator of the notion of reciprocity between form and force

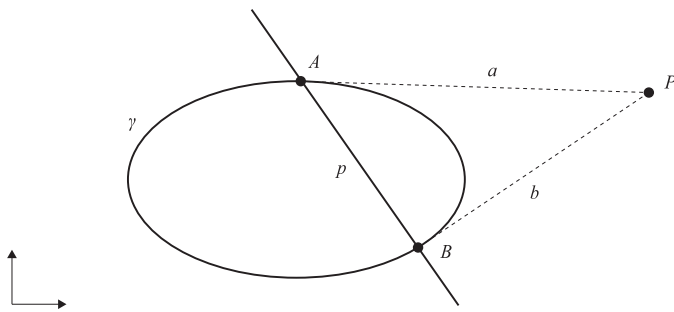


Fig. 2. Polarity on the plane between a point (P) and a line (p) induced by a conic in 2D projective geometry.

diagrams (Charlton, 1982; Kurrer, 2008; Zalewski and Allen, 1998) and who suggested a methodology to construct the diagrams within the context of projective geometry. Relevant work in this direction was also undertaken by Cremona (1872), who introduced an approach alternative to the one of Maxwell for the construction of reciprocal diagrams.

The design and analysis procedures of graphic statics were particularly relevant in the late 19th century, when they were regarded, along with the notions of projective geometry, as essential elements in the education of young engineers (Charlton, 1982). At that time, graphic statics and more generally graphical analysis were used as the conventional approach for the solution of several engineering problems, including safety assessment of masonry vaults, arches and domes. However, most of the applications were largely limited to the standardized analysis of two-dimensional trusses and required a considerable number of hours of draughtsmanship. In the early 20th century, the interest in and research on graphic statics went through a rapid decline. The developments in analytical statics led engineers away from the subject, where no significant progress was made after the late 19th century (Kurrer, 2008). By the second quarter of 20th century, graphic statics was almost entirely replaced by the analytical approaches of elasticity theory, which relied on the solution of equations rather than the construction of time-consuming hand drawing. The fast advances in analytical statics and the introduction of numerical methods such as the finite element method in the second half of the 20th century, led to the widespread situation where in the curricula of young engineers and architects, the study of geometry, let alone projective geometry and its relation to graphic statics gradually declined. In the field of theoretical research, an exception is represented by the work of the structural topology group at the University of Montreal on planar and spatial reciprocal diagrams among others. There, following Maxwell's legacy, the application of projective geometry and higher-dimensional duality between stress functions was developed from a general and mathematical point of view within the context of rigidity theory (Crapo, 1979; Crapo and Whiteley, 1994). In the field of applied research, graphic statics has been proposed, within the domain of plasticity theory, for the design of reinforced concrete structures (Muttoni et al., 1996).

Over the last few decades, graphic statics has seen a renewed interest. This directly reflects the increasing demand for the construction of material efficient structures, which in turn calls for the use of synthetic and intuitive tools that allow the designers to take advantage of the relationship between form and forces from the early stages of the design process. In this context, the tendency of reintroducing graphic statics within the engineering and architectural education is significant (Zalewski and Allen, 1998, 2009; Muttoni, 2011). Recent applications of graphic statics are underpinned by the computational and visualization capabilities offered by contemporary computer aided design tools. These can be

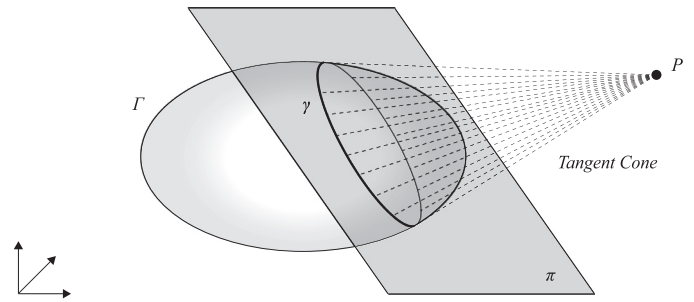


Fig. 3. Polarity in three-space between a point (P) and a plane (π) induced by a quadric in 3D projective geometry.

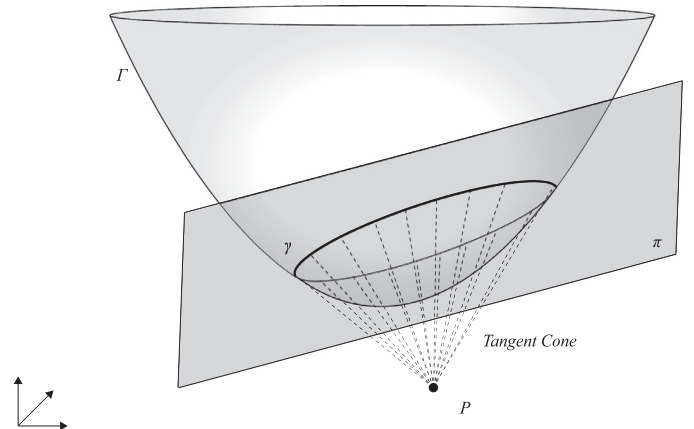


Fig. 4. Polarity in three-space induced by a paraboloid of revolution.

found, among others, in the design of compression-only or tension-only spatial funicular structures by means of the Thrust Network Analysis (TNA) (Block and Ochsendorf, 2007), in the form-finding of tension-compression spatial networks with the Combinatorial Equilibrium Modelling (CEM) (Ohlbrock et al., 2016), and in the analysis and design of compression-only and tension-only spatial polyhedral structures (Akbarzadeh et al., 2015). Moreover, graphic statics can be also used in the context of transformations of form diagrams in static equilibrium (Fivet, 2016). Other recent contributions in the study and generalization of graphical analysis include publications from Fraternali and Carpentieri (014), Micheletti (2008), and Zanni and Pennock (2009). Graphic statics has also regained popularity in the engineering industry as a design and optimisation tool (Beghini et al., 2013; Mazurek et al., 2016). Most of the current graphic statics approaches for the construction and transformation of 3D form and force diagrams rely on the use of iterative algorithms or procedural reconstruction techniques, such as the *polyhedral* one (Akbarzadeh et al., 2015 and Lee et al., 2018), the *vector-based* one (D'Acunto et al., 2017), and operate on a local node-by-node basis. A direct global implementation, which is grounded on the definition of higher dimensional reciprocal stress functions as a tool for generating pairs of reciprocal form and force diagrams, has been also suggested (McRobie, 2016; Konstantatou and McRobie, 2016). This can be directly related to the graphical analysis of general planar and spatial frames using Clifford algebra as well as faces of zero-oriented area (McRobie, 2017). The mathematical toolkit of Clifford algebra, such as bivectors and wedge products, offers a rich geometrical description which can encompass more general reciprocal diagrams where the force faces need not be perpendicular to the form edges. This enables the evaluation of all six stress-resultants rather than solely the axial force even for cases where the geometry of adjacent force faces is not identical.

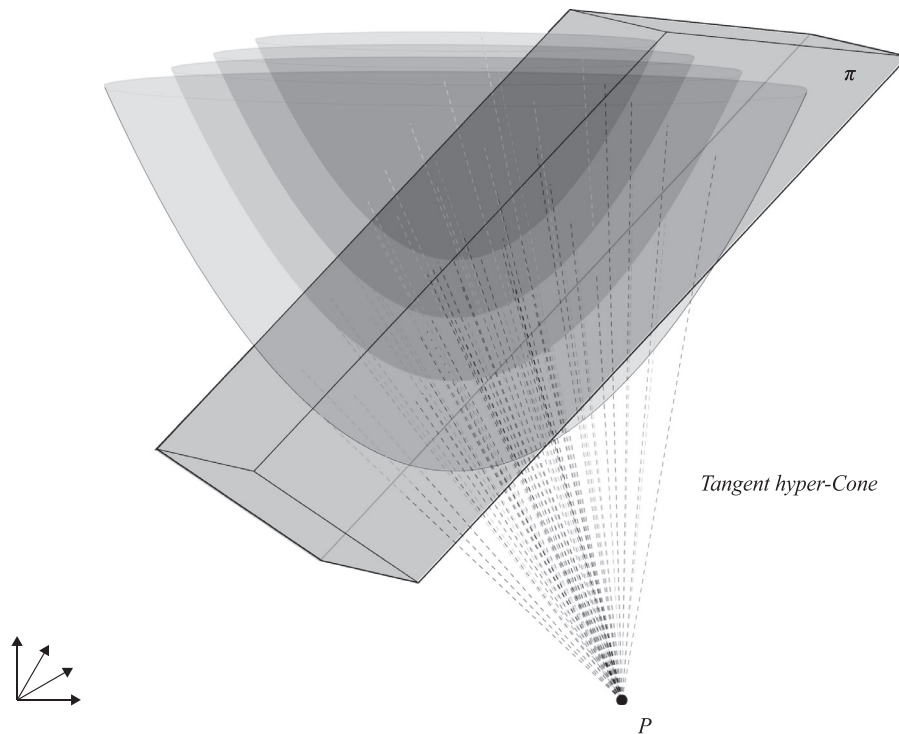


Fig. 5. Polarity in four-space induced by a hyper-paraboloid of revolution (in a schematic representation here).

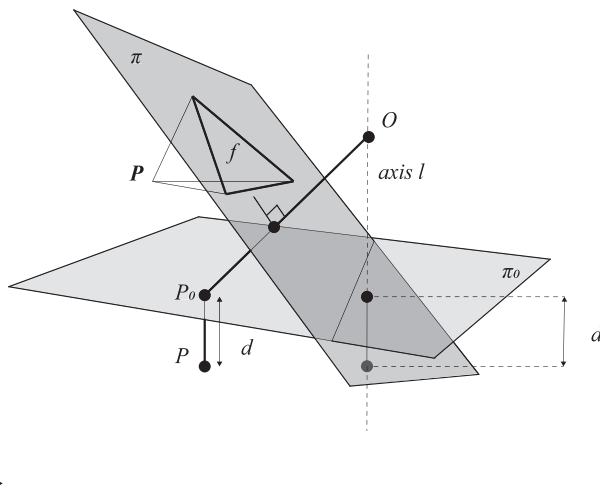


Fig. 6. Plane and origin construction mapping a point (P) and its reciprocal polar plane (π).

2.2. Maxwell, analogies, and his abstract mathematical view of graphic statics

Maxwell was interested in the concept of analogies in science from a very young age. He was particularly intrigued by the relation between beautiful forms and mathematics (Harman, 2004); from how the underlying geometrical principles can act as a morphological mould. That idea was explored conceptually in his essay ‘Analogies in Nature’, which he wrote for the student society ‘Cambridge Apostles’ in 1856 (Harman, 1990). In this work, he mentioned reciprocity which would be a key idea in his seminal papers on structures and electromagnetism (Harman, 1990). Also, few months earlier, he published one of his seminal papers ‘On Faraday’s lines of force’ (Maxwell, 1855) where he related magnetic forces with electric currents through a geometrical construction of

lines of force which run on a surface in an analogous way to how incompressible fluids flow (Harman, 2004). Maxwell’s fascination with analogies and reciprocity was heavily influenced from his contemporaries and took form through the mathematical theories of his time.

The mathematics of late 19th century were characterized not only from projective geometry, the ‘modern geometry’ of the time, but also from the newly formed topology. Maxwell was familiar with the work of Carnot, Chasles, and Poncelet with respect to the former and the work of Gauss, Listing, Euler, Riemann, Leibniz, and Cayley with regards to the latter (Harman, 1998; Maxwell, 1870). His mathematical physics framework was heavily influenced by Gauss’ ‘Geometria Situs’ and Listing’s topology, which shaped his ‘knotted curves’ discussion, his theory on translations and rotations of spatial motions, and fed his electromagnetic analogies and relations between electrical circuits and lines of force (Harman, 1998), (Maxwell, 1855). Moreover, Maxwell also used projective geometry as a tool of geometrical analogy; in his point of view, projective geometry could express the supremacy of the geometrical over the analytical way of thinking (Harman, 1998). This can be seen clearly in his theory of reciprocity between geometrical elements if placed within a projective geometry framework. Maxwell wrote on this: ‘The study of corresponding elements in two figures has led to the establishment of a geometry of position by which results are obtained by pure reasoning without calculation, the verification of which by the Cartesian analysis would fill many pages with symbols.’ (Harman, 1998). As a result, projective geometry and its underlying notion of duality was the tool used by Maxwell to express the reciprocity between not only form and force diagrams in the context of graphical analysis but also, of ‘geometrical optics, electrical circuits, and the kinetic theory of gases’ (Harman, 1998).

Maxwell’s graphical analysis of trusses (Maxwell, 1864, 1867, 1870) was particularly influenced by Chasles’, Monge’s, and Poncelet’s pole and polar construction and duality principle, which expressed the reciprocity between form and force diagrams. As he characteristically put it in 1873: ‘[...] principle of duality...the

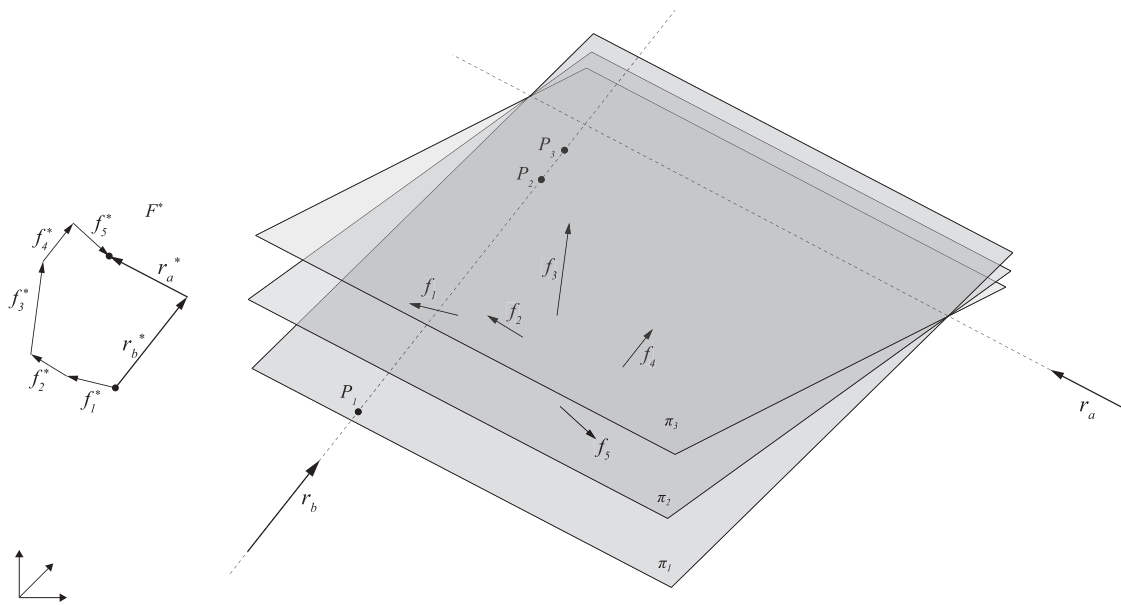


Fig. 7. Given a set of five input forces in space (f_1 to f_5), this can be composed into a pair of skew resultants (r_a and r_b); the lines of action of these resultants are conjugate lines under a null polarity. Given any point on the line of action of one resultant, its polar plane is the plane incident that point and containing the line of action of the other resultant. As such, if a point moves along the line of action of one resultant, its polar plane rotates around the line of action of the other resultant.

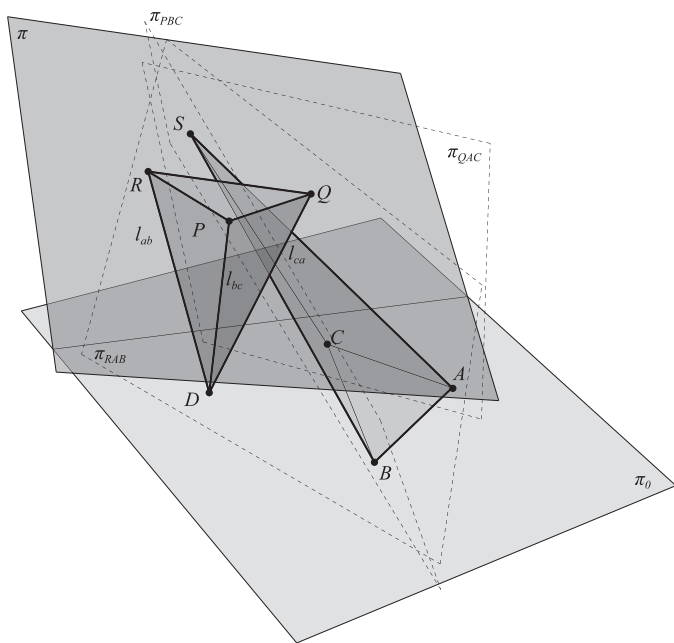


Fig. 8. Null polarity mapping a self-conjugate point (P) and its reciprocal polar plane (π) and the construction of a Möbius pair of tetrahedra.

leading idea of modern geometry’ (Harman, 1998). Maxwell combined the projective geometry duality with Euler’s work on polyhedral counting to develop his theory of reciprocal diagrams in statics. He observed that planar reciprocal diagrams follow the counting rules of polyhedra when it comes to their constituent geometrical elements (points, edges, faces) and that a planar form diagram has a force reciprocal when it is a projection of a polyhedron (Maxwell, 1864). At that point, he would derive reciprocal diagrams by using geometrical constructions with circles, for the planar, and spheres, for the spatial case, which would ensure that corresponding reciprocal elements were perpendicular. However, it is not until few years later that he placed his constructions

one dimension up, and entirely in the space of the reciprocal polyhedra following Poncelet’s and Monge’s theory of polar figures (Charlton, 1982). Maxwell was also familiar with the work of Airy (1862) since he was keeping a regular communication with the Astronomer Royal on all sorts of scientific matters and he was a reviewer for some of his work. In particular, Maxwell along with Rankine were the reviewers of Airy (1862) paper on structural analysis and the Airy stress function (Harman, 1990) which he considered as a significant simplification of the already existing theories and he later developed and applied himself in his graphical analysis of trusses. A number of his contemporaries, (such as Cremona, Rankine, and Culmann) adopted, generalized, or standardized his methods. However, most of them did not grasp or consider reciprocal diagrams in the deep geometrical way Maxwell did. An exception was Cremona, who as a geometer, was also familiar with polarities, and who discussed and applied Möbius’ null-system methods. As a result, Maxwell is credited with making the biggest intellectual contribution in the context of graphic statics (Kurrer, 2008).

2.3. Reciprocal diagrams and stress functions

The introduction of the notion of reciprocal form and force diagrams was attributed to Maxwell by Cremona (1872) for the 2D case and to Rankine by Maxwell (1864) for the 3D case. In his seminal publications on structural theory Maxwell observed that a 2D form diagram in static equilibrium has a reciprocal force diagram if and only if they are projections of equally reciprocal, plane-faced three-dimensional polyhedra (Maxwell, 1864, 1870). As mentioned by Maxwell, each of these reciprocal polyhedra can be constructed from the other one, through a polar transformation based on a paraboloid of revolution (Maxwell, 1870). Moreover, the polyhedron corresponding to a given 2D truss represents an instance of a piecewise linear Airy stress function, with the change of slope between adjacent faces providing the axial loads of the corresponding truss.

In fact, the roles of reciprocal form and force diagrams are interchangeable and there is no distinction between lines of action of the applied forces and structural members. This has been explained in detail in (McRobie et al., 2016) and in (Mitchell et al.,

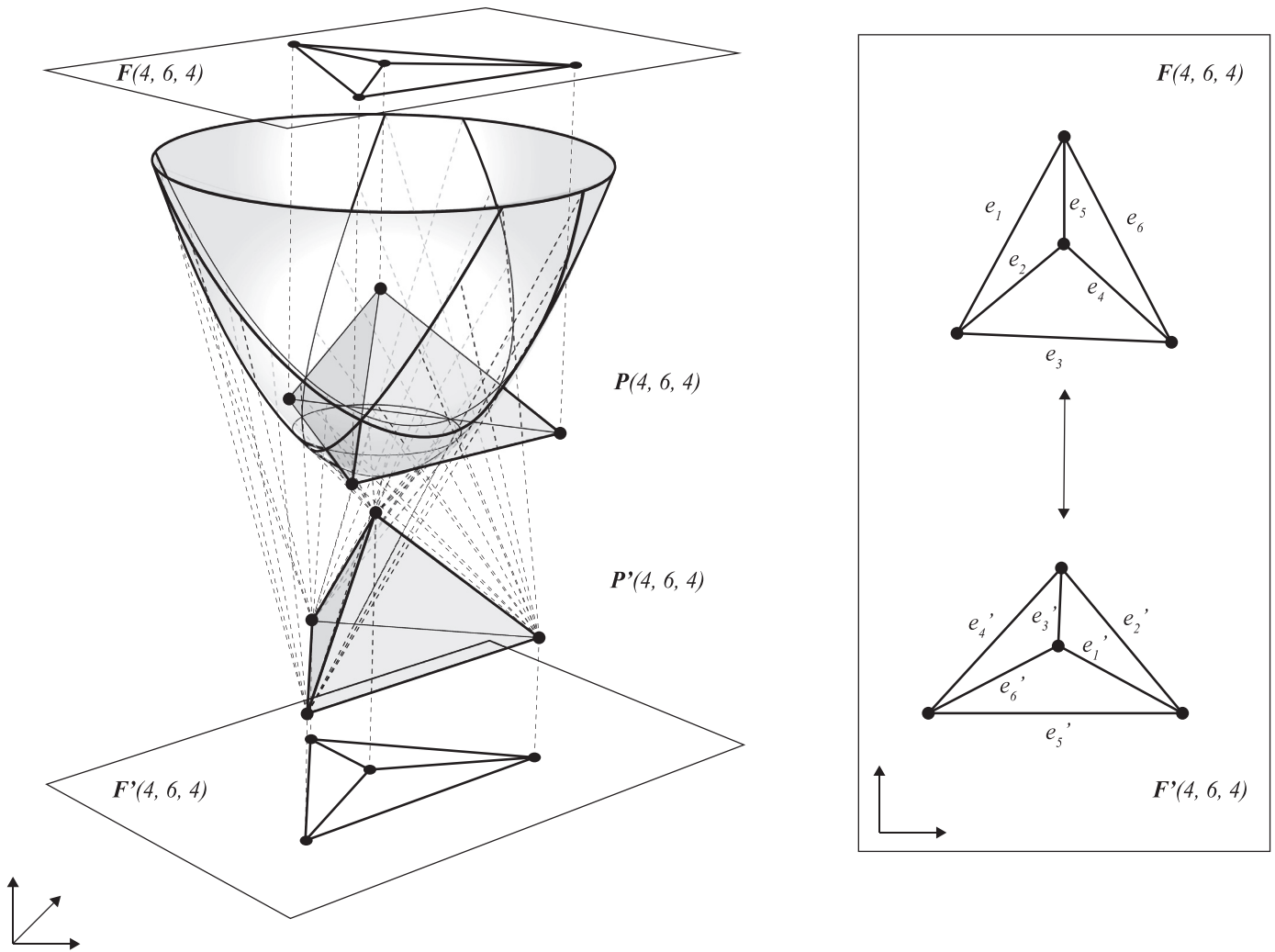


Fig. 9. Left: Reciprocal polyhedral Airy stress functions (\mathbf{P}, \mathbf{P}') through a polarity induced by a paraboloid of revolution. Right: The orthographic projections of the reciprocal polyhedra produce a pair of reciprocal form and force diagrams (\mathbf{F}, \mathbf{F}').

2016) where external forces, which can be applied on the structural perimeter as well as on internal nodes,¹ can be combined with the form diagram to make a projection of a single polyhedron, possibly with the use of an extra portion of a funicular polygon. Thus, this polyhedron can be also seen as the Airy stress function of an equivalent self-stressed truss (Fig. 15). As a result, in the following there is no distinction between these two cases (self-stressed, with external loading) since they are geometrically equivalent.

Depending on their spatial dimension and the geometrical relationship between corresponding members, reciprocal diagrams can be categorised as follows:

- *Maxwell 2D reciprocals*, where edges in the 2D form diagram correspond to perpendicular edges in the reciprocal 2D force diagram;
- *Cremona 2D reciprocals*, where corresponding edges between 2D form and force diagrams are parallel;

- *Rankine 3D reciprocals*, where form edges correspond to reciprocal perpendicular force faces;
- *Cremona 3D reciprocals*, where edges in the 3D form diagram correspond to parallel edges in the reciprocal 3D force diagram.

We should note that Maxwell 2D and Cremona 2D diagrams are identical up to a 90° rotation, whereas Rankine 3D and Cremona 3D are geometrically distinct since in the former forces are expressed by surface areas and in the latter by edge lengths.

2.4. Simply connected n -polytopes

The necessary and sufficient condition for the existence of reciprocal form and force diagrams introduced by Maxwell is equivalent to the independent conditions that a form diagram has an underlying planar graph and it also possess a state of self-stress (Crapo, 1979). A polyhedron can be defined as a set of vertices v , edges e , and faces f : $\mathbf{P}(v, e, f)$. A planar graph $\mathbf{G}(v, e)$ is a set of vertices v and edges e , which can be drawn topologically on a 2D Euclidean plane without any edge crossing. This can be extended to spatial trusses, which are projections of 4-polytopes and have an underlying spatial planar graph. It should be noted that 4-polytopes are the equivalent of polyhedra in the four-dimensional space and are a set of vertices v , edges e , faces f , and cells c : $\mathbf{P}(v, e, f, c)$. The cells lie on hyper-planes

¹ In the general case, the application of an external load to an internal node can result in a non-planar graph (where edges intersect) which is thus not the projection of a simply-connected polyhedron. This issue can be addressed with the inclusion of extra nodes where the line of action of the external load intersects the form edges following Bow's technique (Bow, 1873).

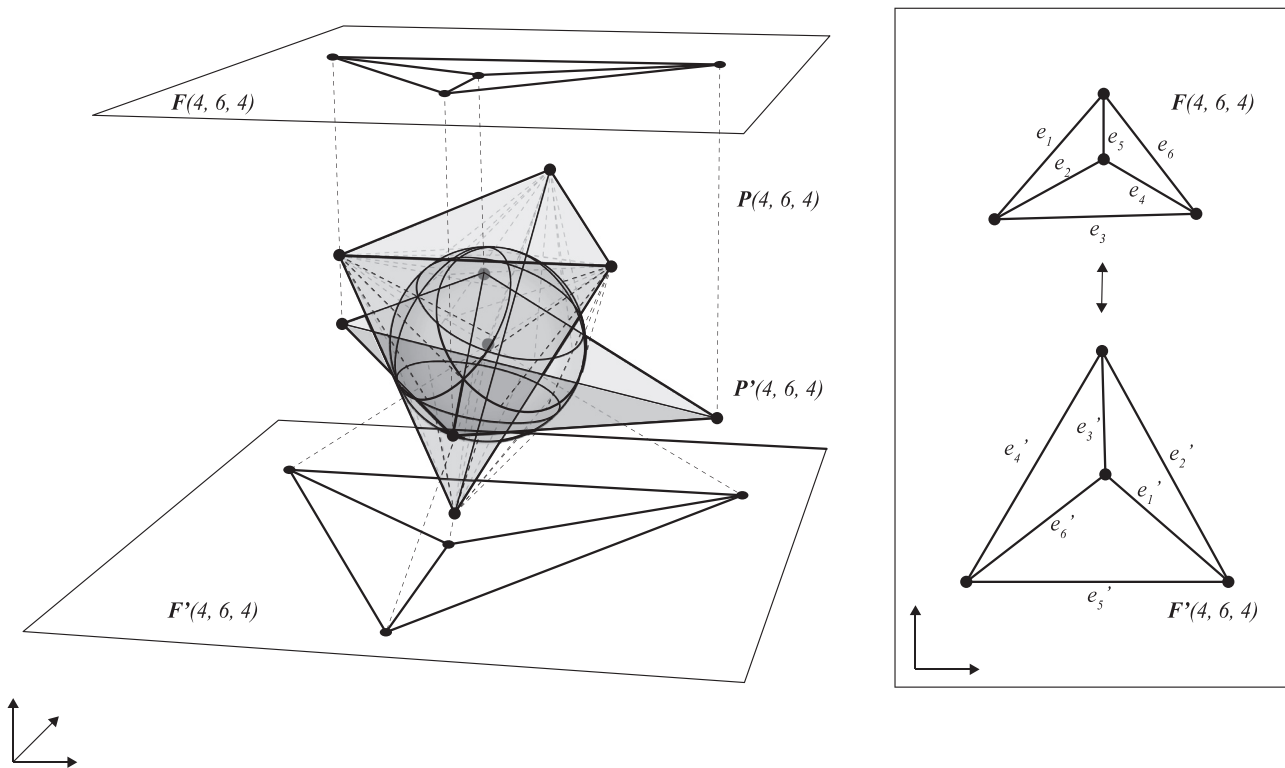


Fig. 10. Left: Constructing a reciprocal polyhedron (P') through a polarity induced by a sphere. Right: The perspective projection from the centre of the sphere of the polar polyhedron together with the orthographic projection of the original polyhedron produce a pair of reciprocal form and force diagrams (F, F').

in the same way that faces lie on planes; hyper-planes constitute three-dimensional subspaces in four-dimensional space in the same way that planes constitute two-dimensional sub-spaces in three-dimensional space. A spatial planar graph $G(v, e, f)$ is an un-oriented graph, which can be drawn topologically in a 3D Euclidean space without any edges or faces crossing (Crapo, 1979). The above statements imply that the polyhedra or 4-polytopes, of which the 2D or 3D trusses are projections, are spherical from a topological standpoint (Crapo, 1979). A topologically spherical, or simply connected, polyhedron $P(v, e, f)$ follows Euler's formula: $v - e + f = 2$: every edge belongs to two faces only and there are no isolated vertices, edges, or faces. Furthermore, the underlying planar graph must have more than three vertices, which should be 3-vertex connected (i.e. if any two vertices are removed and their adjacent edges the graph does not divide into two subgraphs) (Whiteley, 1979). Equally, a simply connected 4-polytope obeys Schläfli's equation, which is a generalization of Euler's formula for n -polytopes (Coxeter, 1973): $v - e + f - c = 0$. Each of its faces belongs to only two cells, every edge is the intersection of at least 3 faces and there are no free vertices, edges, faces, or cells.

The condition of possessing a planar graph is necessary but not sufficient for a 2D truss to have a reciprocal. Together with the additional requirement that the 2D truss has a state of self-stress, these conditions imply that the 2D truss is an accurate picture on the Euclidean plane of a projection of the corresponding polyhedron (Crapo, 1979). As a result, a 2D truss without external loading, which is a projection of a polyhedron, is necessarily self-stressed (Micheletti, 2008). At the same time, given a diagram of a 2D truss with a planar graph the question arises of how to assess whether this diagram corresponds to a projection of a polyhedron. This has been studied in the fields of scene analysis and rigidity theory among others. One of the geometrical algorithms in Euclidean geometry that can be used is based on the fact that every three planes meet at a point (unless they are parallel) and

all the vertices of a face lie on the same plane (Whiteley, 1979). The same algorithm can be generalised for 4-polytopes and hyperplanes; in which case every four hyperplanes meet at a point (unless they are parallel), all vertices of a cell lie on the same hyperplane, and all vertices of a face on the same plane. It should be noted that in the 2D case there can be trusses which do not have a planar graph but nevertheless possess a state of self-stress: it has been proven that such diagrams have equivalent (isomorphic) spaces of internal forces to the ones resulting from adding extra vertices to the points of intersection of crossing edges (Bow, 1873; Crapo and Whiteley, 1993). The final diagram, with the possible addition of vertices, is self-stressed, has a planar graph and is thus a projection of a polyhedron. However, while this construction does not apply to the 3D case, it should be noticed that for spatial trusses, cases can be found where trusses are not projections of simply connected 4-polytopes and nonetheless, they do possess states of self-stress. In contemporary nomenclature this is called 'Rankine incompleteness' and has been discussed recently in the graphic statics literature (McRobie, 2016, 2017).

The polarities approach followed here can be applied to any 2D self-stressed truss, to any 3D self-stressed truss which is a projection of a simply-connected 4-polytope, and to rotationally symmetric spatial self-stressed structures (such as tensegrities) through the technique of coning. Following this method, spatial structures which have just one cell, which would otherwise map to just a single point, can be further subdivided to numerous cells which in turn can produce a geometrically richer force reciprocal (McRobie, 2017).

2.5. Projective geometry and principle of duality

Projective geometry is a system of geometric prepositions that does not include any of the familiar Euclidean notions of parallelism, intermediacy, angle and length measurement. Its origins

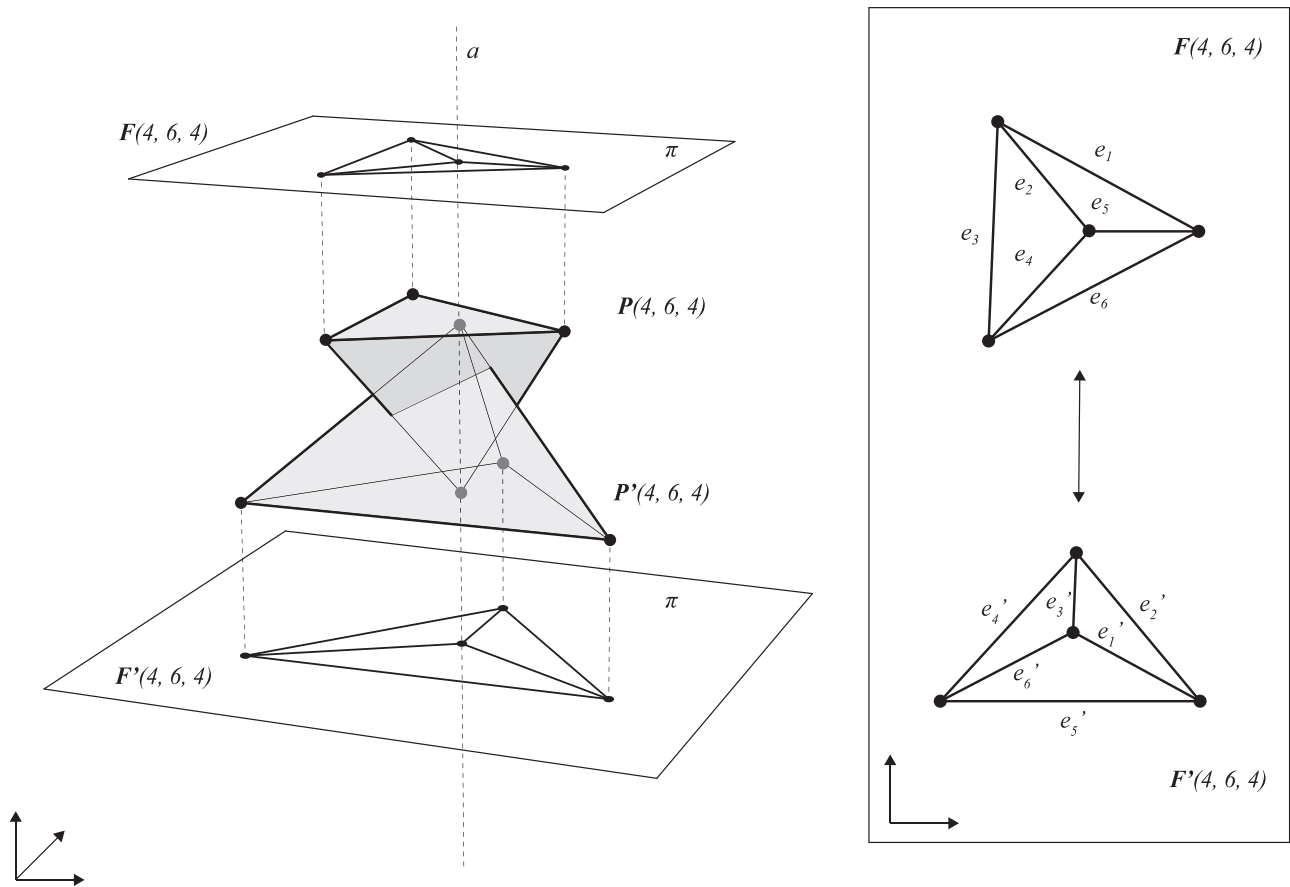


Fig. 11. Left: Reciprocal polyhedral Airy stress functions (\mathbf{P}, \mathbf{P}') through a null-polarity. Right: The orthographic projections of the reciprocal polyhedra along the central axis produce a pair of Cremona 2D reciprocal form and force diagrams (\mathbf{F}, \mathbf{F}').

can be traced back to Pappus of Alexandria (4th century) as well as the French architect Desargues (16th century) and the French philosopher and mathematician Pascal (17th century) (Coxeter, 1969; Rosenfeld, 1988). On the projective plane, two lines always meet in a point, either an ordinary point or a point at infinity (Coxeter, 1974). The set of all the points at infinity constitutes the line at infinity, which together with the Euclidean plane defines the projective plane. On the projective plane, there is only one type of conic, or conic section; it is its position with respect to the line at infinity that gives rise to the different embeddings (circle, ellipse, parabola, hyperbola) when seen from a Euclidean point of view (Coxeter, 1974; Rosenfeld, 1988). Moreover, thanks to the principle of duality, any projective geometry proposition that is true for points and lines can be dualized to an equivalently true proposition for lines and points. The principle of duality thus works by interchanging primitive geometrical elements of the projective plane. This can be extended to higher spatial dimensions by taking into account the primitive elements of the specific projective space under consideration (i.e. the point and the line in 2D; the point, the line and the plane in 3D; the point, the line, the plane and the hyperplane in 4D).

Considering that an n -dimensional form diagram and its reciprocal force diagram are projections of $(n+1)$ -dimensional stress functions, this reciprocal pair also obeys the counting rules and connectivity of their higher-dimensional stress functions (Fig. 1). For example, a 2D truss and its 2D force reciprocal are projections of 3D polyhedral stress functions and they thus obey the connectivity and counting rules of these polyhedra: points map to planes, lines to lines, with the connectivity remaining fixed; if two points of the form diagram are connected via an edge then the

corresponding faces of the force reciprocal intersect and share a common edge.

2.6. Polarities

A polarity, or polar transformation, is a correspondence which maps geometrical elements of the projective space to each other according to the principle of duality (Fig. 1) and has a degree-2 (transforming an element twice results in the initial element). Desargues was the first one to define the notion of polar transformation on the projective plane (Rosenfeld, 1988), where every point (*pole*) is transformed into a line (*polar line*) through a conic and vice versa. Subsequently, Monge proposed a generalization of Desargues' polarity to the projective three-space, where every point (*pole*) is mapped to a plane (*polar plane*) through a quadric (Monge, 1794) and vice versa. We should note that the quadric is the generalisation of the conic in three-space: a surface (such as the sphere, the ellipsoid, the hyperboloid, and the paraboloid) described in Euclidean geometry by a polynomial of degree two. Monge used a paraboloid of revolution as the base quadric of the polarity to derive reciprocal polyhedra (Chasles, 1875) in the same way as mentioned by Maxwell in his 1870 paper. The work of Monge was then systematized and further developed by his student Poncelet who formalised the principle of duality (Poncelet, 1822; Rosenfeld, 1988; Cremona, 1873). From an algebraic standpoint, significant contributions to the field were made by Möbius, who showed that projective transformations and polarities can be treated in an analytic way (Rosenfeld, 1988), representing them in matrix form (Gray, 2010) (see Section 2.5.3). Grounded on the principle of duality and linear algebra, Möbius developed the

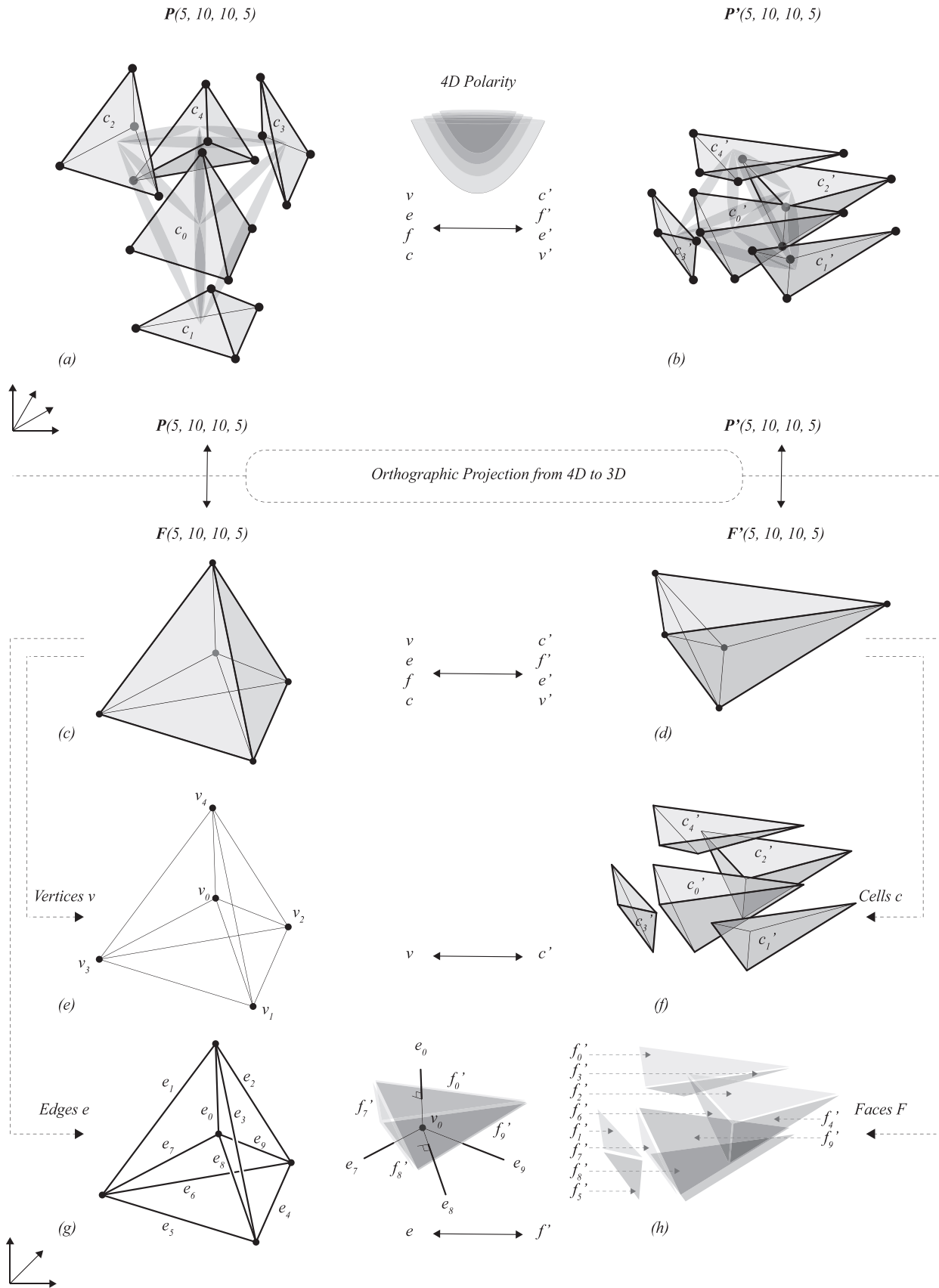


Fig. 12. (a, b): Topology of two reciprocal 4-polytopes (P, P'), constructed through a polarity induced by a hyper-paraboloid of revolution; (c, d): The orthographic projection of the 4-polytopes produces a pair of reciprocal form and force diagrams (F, F') which follow 4D duality between their geometrical elements; (e, f): form vertices v map to reciprocal cells c ; (g, h): form edges e map to reciprocal and perpendicular force faces f .

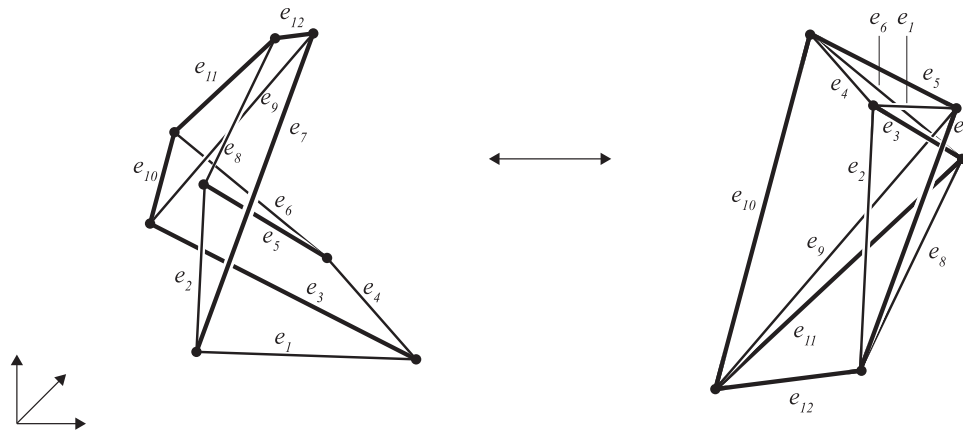


Fig. 13. Cremona 3D reciprocals. Left: Spatial form diagram which is topologically a cube. Right: Reciprocal force diagram which is topologically an octahedron with corresponding edges parallel.

concept of null-polarity, a form of polarity that is not related to any quadric and can be defined only for odd-dimensions (Baer, 1945; Gray, 2010). The work of Möbius had a significant role in the field of graphic statics thanks to Cremona, who adopted his null-polarity in 3D to obtain Cremona 2D reciprocals, albeit in a geometrical rather than algebraic way.

Polarities, not only are of degree-2, where the image of a transformed element is the element itself (e.g. in the projective plane, a point A maps to a line a which maps back to point A) but also, they preserve incidence (the abstract concept which expresses whether two objects ‘contain’ each other: if a point is incident to a line then the point lies on that line; if a line is incident to a plane then the line belongs to that plane; if two lines intersect in a point, this point is incident to both lines and consequently these two lines are concurrent. A 2D example of how incidence and duality are preserved through polarities could be: on the projective plane, if three lines a, b, c are concurrent on a point D , then the polar points A, B, C lie on the polar line d) (Coxeter, 1969). Polarities can be categorised depending on whether and how they admit self-conjugate points (i.e. points that lie on their polar lines on the projective plane or on their polar plane in the projective three-space). On the projective plane, polarities can be of two types: *hyperbolic* (with self-conjugate points that, as observed by Von Staudt (Coxeter, 1969), are the locus of a conic), and *elliptic* (which does not admit any self-conjugate point). In the projective three-space there are four types of polarities: the *elliptic* polarity (without self-conjugate points), two different *hyperbolic* polarities (which differ depending on whether the locus of self-conjugate points is a ruled quadric or not) and the *null-polarity* (where every point of space is self-conjugate) (Coxeter, 1998). Since polarities are global transformations mapping every element in space to another they are a useful tool for generating global force reciprocals of structures in static equilibrium as well as for transforming structures from one typology to another.

2.6.1. Construction of hyperbolic polarities in 2D, 3D and 4D space

Hyperbolic polarities can be constructed with simple geometric procedures, as explained by Desargues, Monge, and Poncelet. Given a generic point P (pole) and a conic γ on the projective plane, in such a way that P is outside γ , the two lines a and b incident to P and tangent to γ meet the conic in two intersection points A and B ; the chord through A and B defines the polar line p (Fig. 2). Equivalently, given any pole P outside of a quadric Γ in the projective three-space, the cone with vertex in P and tangent to Γ , intersects the quadric in a planar curve γ (a conic), which lies in the polar plane π (Fig. 3). Equivalent constructions can be

followed when P is inside the conic (Fig. A.1) or quadric (Fig. A.2) (see Appendix A). We should note that these constructions readily apply on the Euclidean plane and space for any embedding of a conic (circle, parabola, ellipse, hyperbola) and real non-degenerate quadric (sphere, paraboloid, ellipsoid, hyperboloid), in this section we use as a 2D example an ellipse and as a 3D example an ellipsoid.

Considering these geometric constructions from a Euclidean perspective and reclaiming the distinction between various non-degenerate conics and quadrics, the relations between dual elements can be expressed in analytical form, using Cartesian coordinates (Smith, 1886). These were already studied in Maxwell’s time in the 19th century. For the general equation of a quadric:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \tag{1}$$

and given a pole $P(x', y', z')$, the corresponding polar plane π is described by the following equation:

$$x(ax' + hy' + gz' + u) + y(hx' + by' + fz' + v) + z(gx' + fy' + cz' + w) + ux' + vy' + wz' + d = 0. \tag{2}$$

Polarities can be thought of as pairs of transformations L, L^{-1} that map a plane π , defined in equation form as $z = Ax + By + C$ and described by the corresponding quadruples $(A, B, 1, C)$, to a point P described from the triple (x', y', z') and vice versa ($L(\pi) = P, L^{-1}(P) = \pi$). Using, for example, a paraboloid of revolution with equation $x^2 + y^2 - 2cz = 0$ as the quadric of the polarity, for a point $P(x', y', z')$ the equation of the polar plane is $z = (x'/c)x + (y'/c)y - z'$, with $L: L(A, B, 1, C) = (cA, cB, -C)$ and $L^{-1}: L^{-1}(x', y', z') = (x'/c, y'/c, 1, -z')$ (Fig. 4).

In the four-dimensional space, following the duality principle described above, a point P is mapped to a polar hyper-plane π . Specifically, given a point P outside a hyper-quadric Γ (i.e. a four-dimensional generalisation of a three-dimensional quadric), the hyper-cone with vertex in P and tangent to Γ , intersects Γ in a quadric that lies on the hyper-plane π (Fig. 5). In equation form, and for the hyper-paraboloid of revolution: $2cw = x^2 + y^2 + z^2$, for a point $P(x', y', z', w')$ and for a plane $\pi(w = Ax + By + Cz + D)$, $L: L(A, B, C, 1, D) = (cA, cB, cC, -D)$ and $L^{-1}: L^{-1}(x', y', z', w') = (x'/c, y'/c, z'/c, 1, -w')$.

An alternative geometric procedure to find the pole P of a given polar plane π in projective three-space is the *plane and origin* construction, described by Maxwell (Maxwell, 1870). Given a polyhedron P , a face f which lies on a plane π , an arbitrary origin O ,

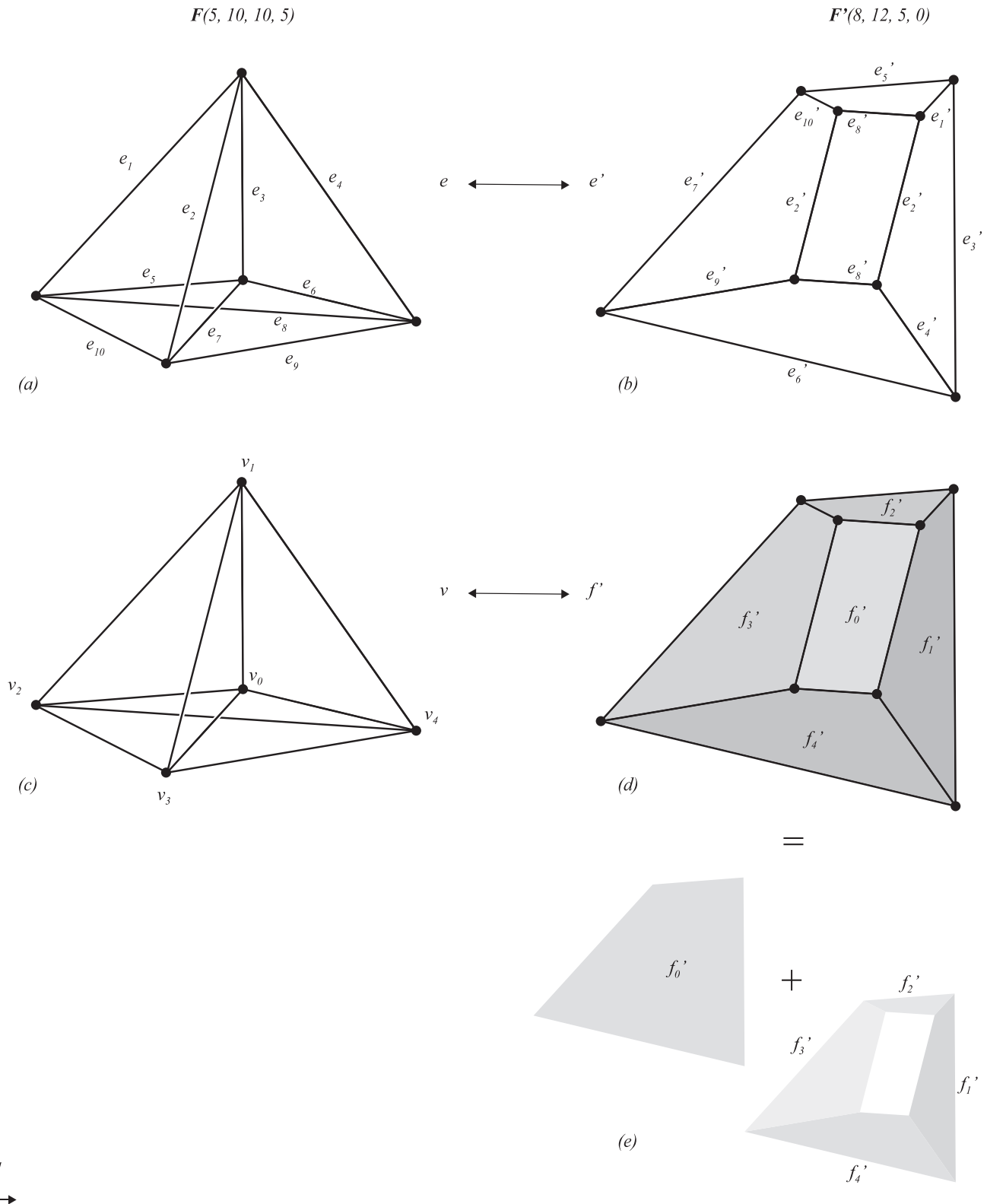


Fig. 14. Vector-based 3D diagrams: (a): 3D form diagram of a spatial truss with a 5-simplex geometry $F(5, 10, 10, 5)$; (b): 3D force diagram with corresponding force edges e' parallel to the form edges e ; (c, d, e): The form vertices v map to force faces f' following a 3D duality, however; the resulting force diagram $F'(8, 12, 5, 0)$ duplicate edges.

and an arbitrary plane π_0 that does not contain O , a line l (axis) through O and perpendicular to π_0 is first fixed. The perpendicular line from O to the given polar plane π intersects π_0 in the point P_0 , which corresponds to the projection of the pole P on the plane π_0 ; the pole P thus lies on the line containing P_0 and perpendic-

ular to π_0 . The position of P can be found considering that the distance between P and P_0 equals the distance d of the intersection of the axis l with the plane π but on the other side of π_0 (Fig. 6).

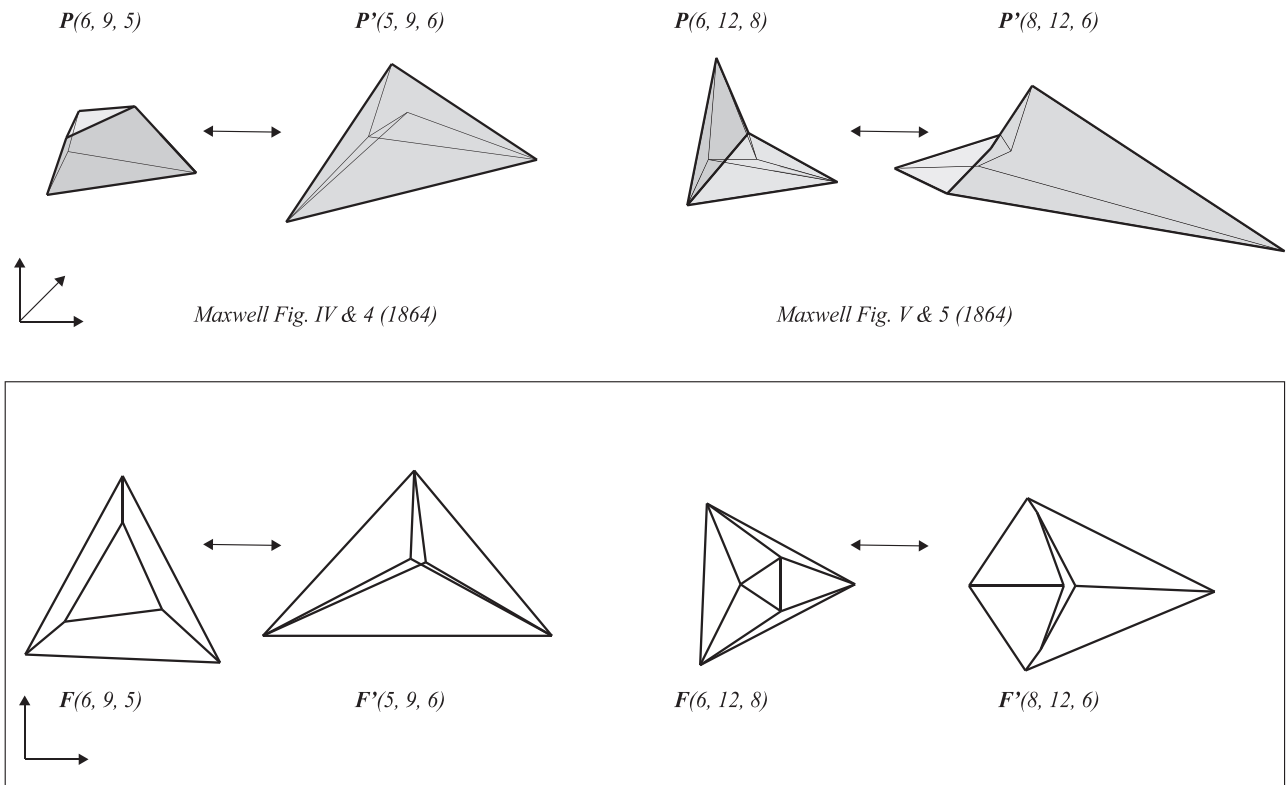


Fig. 15. Reciprocal polyhedral Airy stress functions (P, P') and resulting pairs of form and force diagrams (F, F') in Maxwell 2D configuration for Maxwell's figures IV & 4 (left) and figures V & 5 (right) (1864).

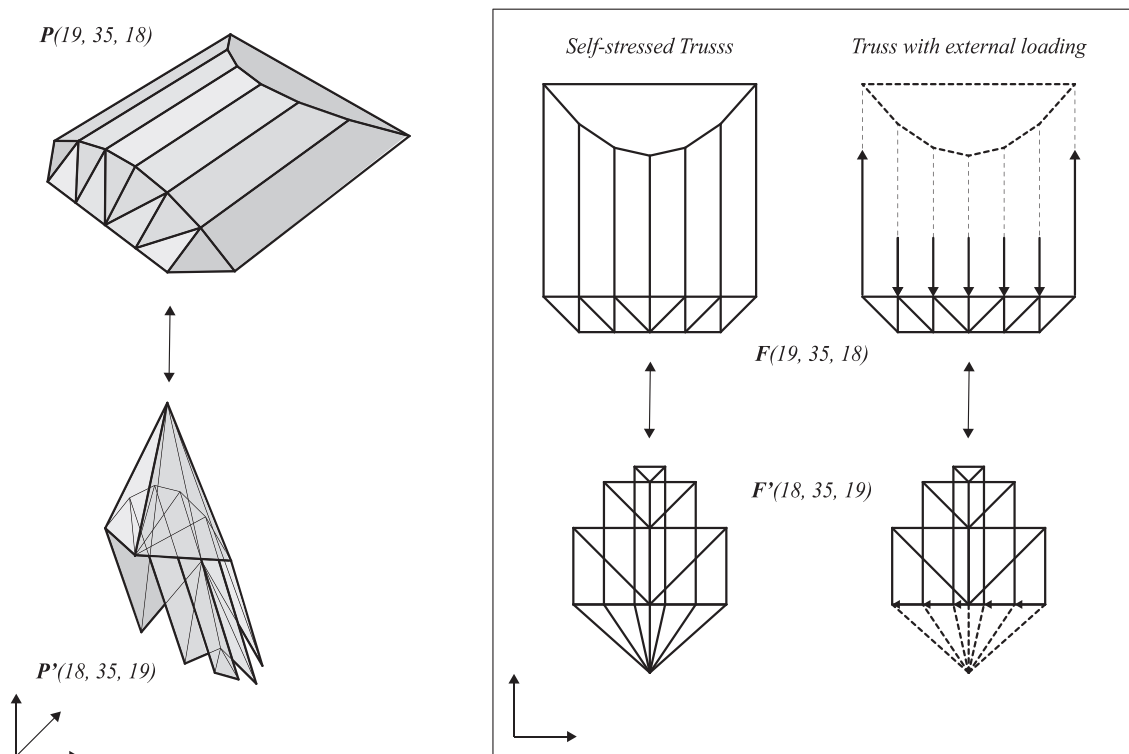


Fig. 16. Reciprocal polyhedral Airy stress functions (P, P') for a Pratt truss (F) (top) and its force reciprocal (F') (bottom). We highlight how the same Airy stress function can correspond to a Pratt truss with external loading (top right) and a self-stressed truss with an equivalent geometry (top middle).

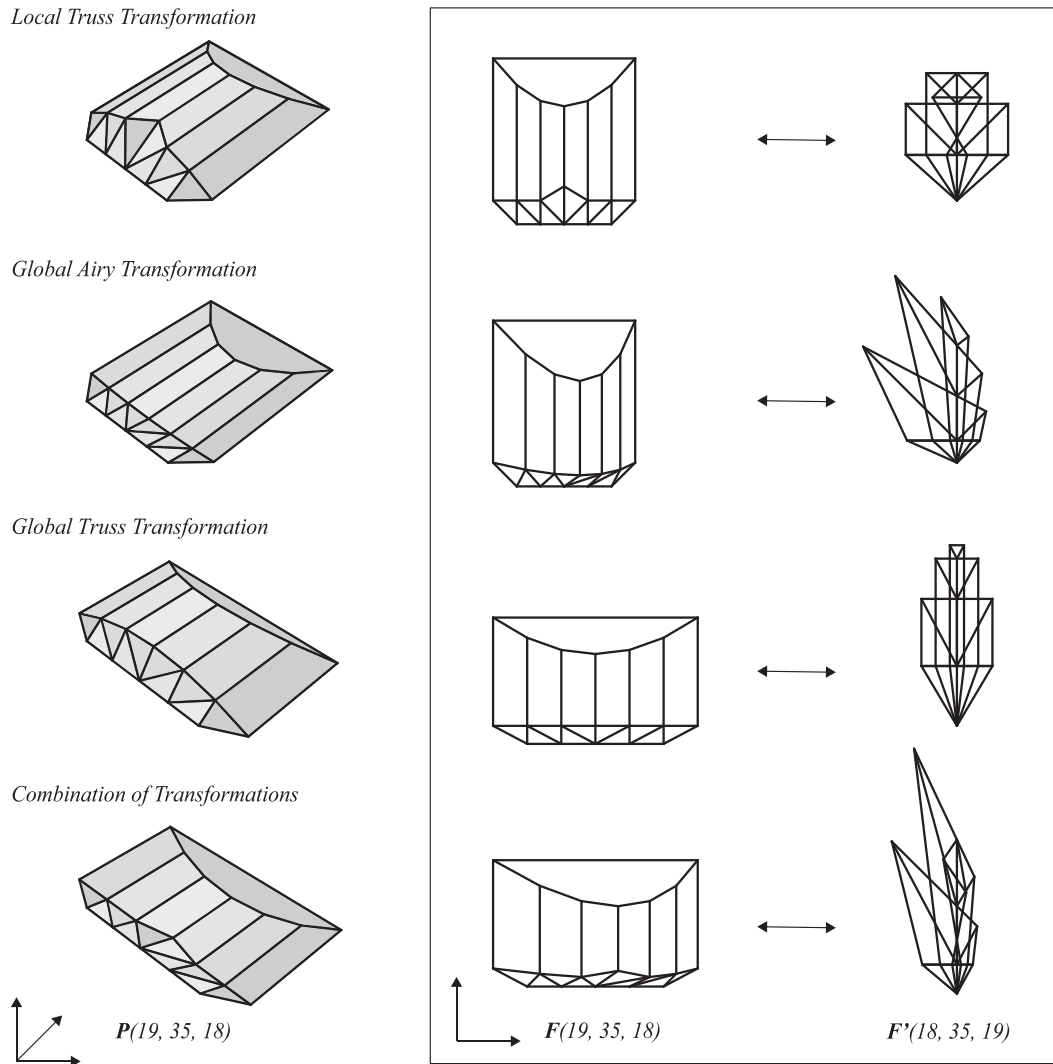


Fig. 17. Polyhedral Airy stress function (P) and planar form and force diagrams (F, F') for a Pratt truss under local and global transformations, from top to bottom: local transformation by moving a truss node; global transformation by shearing the Airy stress function; global transformation by scaling the form diagram; combination of the above transformations. Equivalent, F can be seen as a force diagram and F' as a reciprocal form diagram.

2.6.2. Construction of null-polarity in 3D space

As observed by Möbius (1833), a system of non-equilibrated and non-coplanar forces in space can be composed up to a pair of resultants or, equivalently, up to one resultant and one couple. This can be achieved through various geometric constructions (D'Acunto et al., 2016). In the first case, infinite pairs of resultants can be found for a given system of forces so that the two lines of action of each pair constitute conjugate lines under a null-polarity (Cremona, 1872). As such, regarding a point on one of the two conjugate lines as a pole, its polar plane is the plane incident with that point and containing the other line (Fig. 7). As pointed out by Cremona (1872), for a given system of forces in space, once one of the two conjugate lines is fixed, the other one is univocally determined. One of the main peculiarities of the null-polarity, which differentiates it from elliptic and hyperbolic polarities, is that every point in space is a self-conjugate point (i.e. it lies on its polar plane). Hence, in every point and plane of space it is possible to find a flat pencil of lines (i.e. a set of infinite coplanar lines that are all incident to the same point), the centre of which is a pole lying on the polar plane defined by the pencil (Coxeter, 1998). As a result, for a given system of forces in space once a pole is fixed its polar plane is univocally determined and vice versa.

The lines that form the flat pencil are self-polar lines or null lines (i.e. the polar planes of the points constituting these lines are incident to the lines themselves) and the moment of the given system of forces with respect to any of them is null. The set of all self-polar lines in a null-polarity constitutes a linear complex in the projective three-space (Coxeter, 1998), which is univocally defined by five independent skew lines. It should be noticed that the null-polarity cannot be induced by a quadric (Konstantatou and McRobie, 2016). To construct a null-polarity in the projective three-space (Cremona, 1890) (Fig. 8), for an arbitrary plane π_0 and four points A, B, C, D on it, which every three are not collinear, three arbitrary planes π_a, π_b, π_c , which pass through the lines AD, BD, CD respectively are fixed. The planes π_a, π_b and π_c intersect each other in the lines l_{bc}, l_{ca}, l_{ab} . For any plane π that intersects the lines l_{bc}, l_{ca}, l_{ab} in the points P, Q, R respectively, the planes $\pi_{PBC}, \pi_{QCA}, \pi_{RAB}$ are then defined. These planes are all incident with the same point S , which is the pole of the polar plane π and is self-conjugate since it lies on it. Based on this construction, the two tetrahedra $ABCS$ and $PQRD$ are reciprocal under the null-polarity (Möbius pair of tetrahedra) (Fig. 8): these tetrahedra are mutually inscribed into each other, since each of the vertices (pole) of a tetrahedron lies

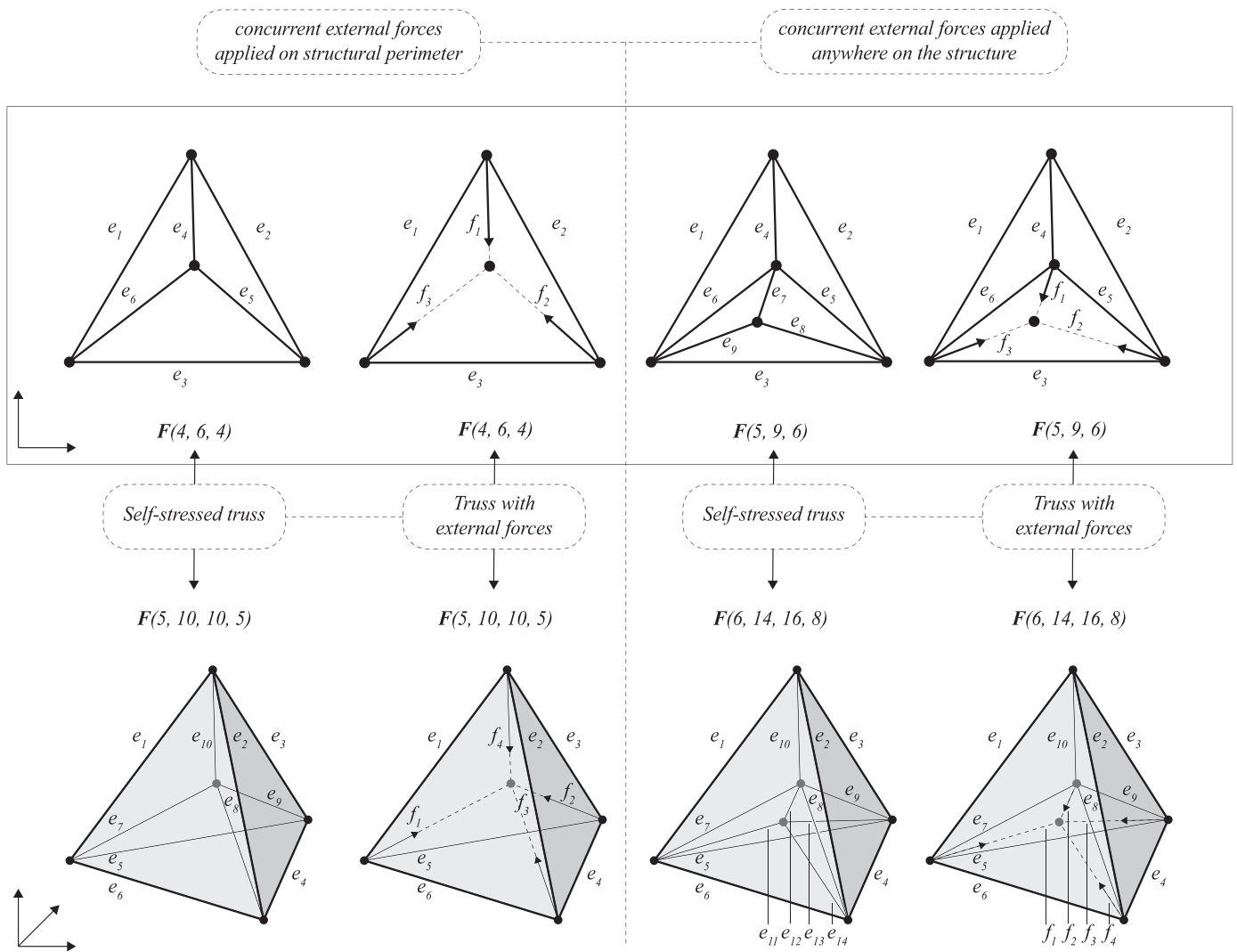


Fig. 18. incorporation of boundary conditions / external forces in the form diagram. Left: adding concurrent forces on the structural boundary for a 2D (top) and 3D (bottom) truss and equivalent self-stressed form diagrams. Right: adding concurrent forces anywhere in the structure.

on one face (polar plane) of the other tetrahedron and vice-versa (Crapo, 1979).

Among the infinite pairs of resultant and couple equivalent to a given system of forces in space, it is possible to define one that is constituted respectively by a force vector and a moment vector that are parallel to each other. The line of action of the resultant represents the *central axis* of the system of forces. Considering the system of forces applied to a free rigid body, the linear complex defined by the null-polarity is perpendicular to a spatial motion of the rigid body, which as known by Chasles (1830), can be represented as the composition of a translation and a rotation around the axis of this screw motion.

2.6.3. Matrix representation of polarities in 3D and 4D space

Polarities and projective geometry in the context of graphic statics have not only been explored from a geometrical point of view but also from an algebraic one; the latter following a matrix approach. In the first case, the geometrical constructions are based on the notion of conics, quadrics, and their higher-dimensional analogues and they apply to both 2D and 3D graphic statics, whereas in the latter case they are based on the notion of null-polarity which applies only to 2D graphic statics (since it cannot be defined in even dimensions). In this research paper both approaches are explored, thereby providing insights into - and a more

holistic view towards - the various polarity-based methods of generating reciprocal diagrams. In particular, it is highlighted how the matrix approach can be used for any polar transformation and how the geometrical approach can be used for the null-polarity. Thus, ultimately the two approaches (projective geometry constructions and matrix transformations) are equivalent with respect to constructing any type (Maxwell 2D, Cremona 2D, Rankine 3D) of reciprocal diagrams.

A non-degenerate quadric, described in its general equation form in (1), can be represented in matrix form (Vaisman, 1997)

as: $\mathbf{x}^t \mathbf{S} \mathbf{x} = 0$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$, and the symmetric matrix associated to the quadric is

$$\mathbf{S} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A point $P = (x_p, y_p, z_p, 1)$ is then mapped to its polar plane π through the polar transformation (in homogeneous coordinates) $P^t \mathbf{S} \mathbf{x} = 0$.

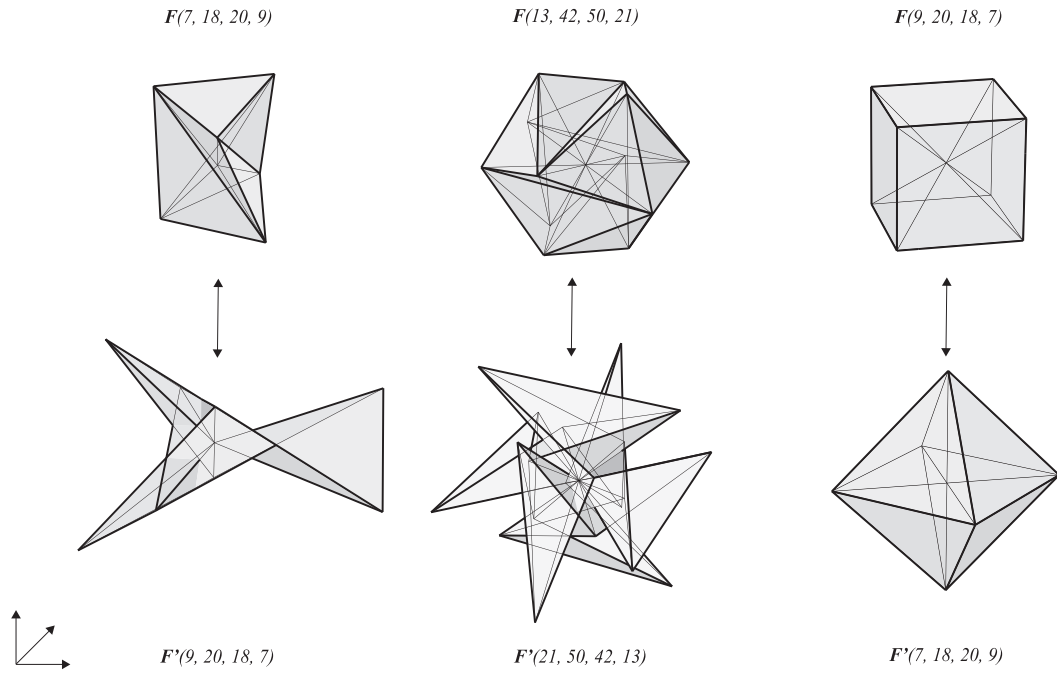


Fig. 19. Pairs of spatial reciprocal diagrams in a Rankine 3D configuration. Left: Tensegrity 3-prism. Middle: Jessen icosahedral tensegrity. Right: Spoked cube.

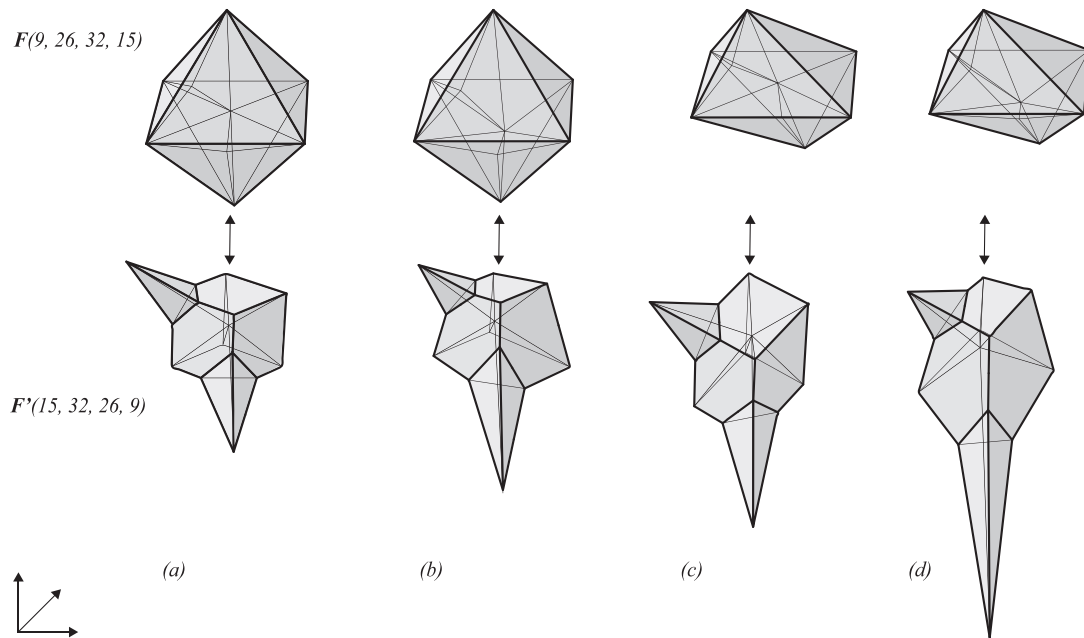


Fig. 20. (a): A polyhedral spatial truss (F) with three states of self-stress (top) and its corresponding Rankine 3D force diagram (F') (bottom) or equivalently a polyhedral spatial truss (F') and a Rankine 3D force diagram (F); (b): Local transformation by moving an internal node of F; (c): Global projective transformations (non-uniform scaling and shearing) applied on F; (d): Combinations of local and global transformations.

For example, considering as the quadric of the polarity a real ellipsoid with canonical equation (in Cartesian coordinates):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

A point $P=(x_p, y_p, z_p)$ is mapped to its polar plane π :

$$\frac{xx_p}{a^2} + \frac{yy_p}{b^2} + \frac{zz_p}{c^2} = 1$$

through the transformation (in homogeneous coordinates):

$$(3) \quad \begin{bmatrix} x_p & y_p & z_p & 1 \end{bmatrix} \begin{bmatrix} 1/a^2 & 0 & 0 & 0 \\ 0 & 1/b^2 & 0 & 0 \\ 0 & 0 & 1/c^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0.$$

(5)

A null-polarity in the projective three-space can be represented in matrix form by an anti-symmetric, (or skew) matrix **A**. In case the central axis coincides with the axis z, the polarity maps a point

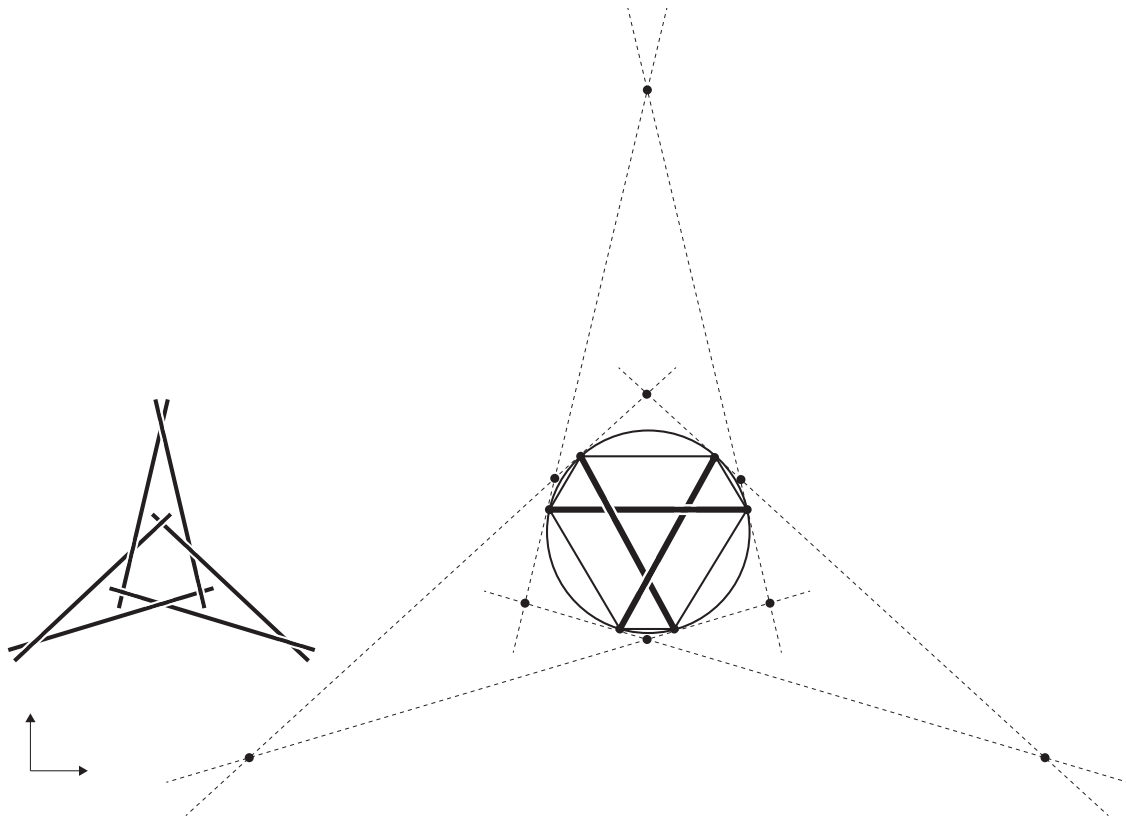


Fig. 21. Polar transformations between planar tensegrity structures (on the right) and grillages (on the left), adapted from Tarnai (1989).

$P=(x_p, y_p, z_p)$ to a polar plane π (Cremona, 1872):

$$xy_p - yx_p + k(z - z_p) = 0 \tag{6}$$

through the transformation (in homogeneous coordinates):

$$\begin{bmatrix} x_p & y_p & z_p & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k \\ 0 & 0 & k & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0. \tag{7}$$

The matrices associated to non-degenerate quadrics and null-polarity can be converted to each other through affine (reflection, scaling, shear, rotation, etc.) or more generally projective transformations using corresponding transformations matrices \mathbf{T} . For instance, in order to convert the matrix \mathbf{S} of a real ellipsoid to the matrix \mathbf{S}' of an elliptic hyperboloid, it is possible to multiply the former with the following transformation matrix:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which corresponds to an affine transformation representing a reflection along the z-axis. As a result, any type of quadric or null-polarity can be used as described above to obtain a reciprocal polar plane for a given point P and then by using a suitable matrix \mathbf{T} convert it to any other polar construction.

The representation of polarities in matrix form can be extended to higher dimensions. In the projective four-space, the matrix form

of a non-degenerate hyper-quadric is: $\mathbf{x}^t \mathbf{S} \mathbf{x} = 0$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \\ 1 \end{bmatrix}$,

and the symmetric matrix associated to the hyper-quadric is

$$\mathbf{S} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}.$$

For example, considering as the quadric of the polarity a 4-elliptic paraboloid, a point $P=(x_p, y_p, z_p, w_p, 1)$ is mapped to its polar plane π :

$$\frac{xx_p}{a^2} + \frac{yy_p}{b^2} + \frac{zz_p}{c^2} - \frac{w}{2} - \frac{w_p}{2} = 0 \tag{8}$$

through the transformation (in homogeneous coordinates):

$$\begin{bmatrix} x_p & y_p & z_p & w_p & 1 \end{bmatrix} \begin{bmatrix} 1/a^2 & 0 & 0 & 0 & 0 \\ 0 & 1/b^2 & 0 & 0 & 0 \\ 0 & 0 & 1/c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \\ 1 \end{bmatrix} = 0.$$

(9)

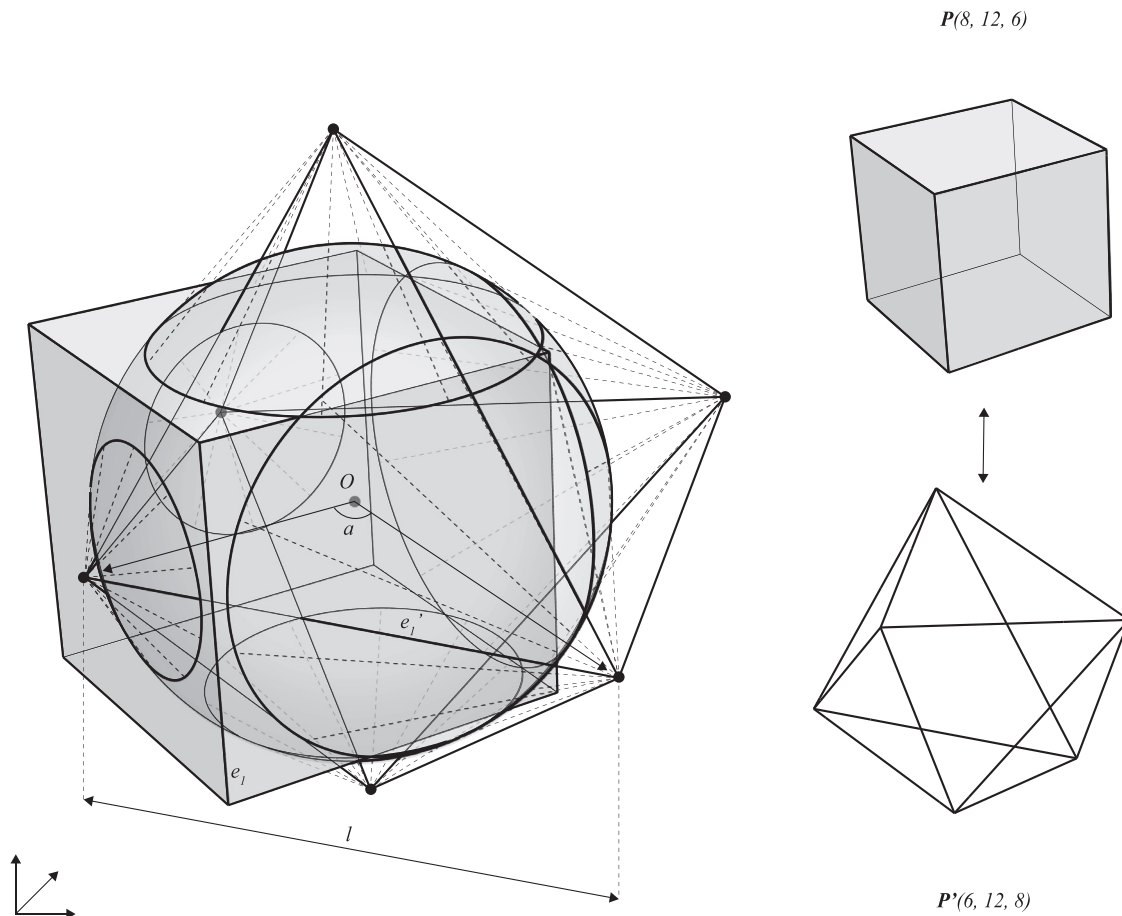


Fig. 22. Reciprocal spatial truss and hinged sheetwork on the right, induced by a polar transformation using a sphere on the left as described in the work of Wester (1989).

3. Construction of reciprocal form and force diagrams

In this section, the theoretical notions and geometric procedures outlined above are applied to structures in global static equilibrium to construct reciprocal form and force diagrams that are projections of higher-dimensional and simply connected stress functions. As already mentioned, the form diagrams need not be tension-only or compression-only and can represent structures which are either self-stressed or under external loading. Reciprocal diagrams in the projective two-space (respectively three-space) can be produced from any polarity induced by a non-degenerate quadric or null-polarity (respectively non-degenerate hyper-quadric) defined in the projective three-space (respectively four-space). For the examples visualised here, the simplest cases are used (i.e. those of the paraboloid of revolution and the sphere). The constructions are developed using the CAD platform *McNeel Rhino*, together with the plug-in the *Grasshopper* and customised *Python* codes.

3.1. Maxwell 2D

3.1.1. Orthographic projections of reciprocal polyhedra induced by a paraboloid of revolution

For a given tetrahedral Airy stress function $\mathbf{P}(v, e, f)$, the planes on which the faces f lie are mapped to their reciprocal vertices v' (Fig. 1) through a paraboloid of revolution (Fig. 9). Considering that the connectivity of the vertices $C_{v'}$ follows the connectivity of the faces C_f , a reciprocal tetrahedron $\mathbf{P}'(v', e', f')$ can be constructed. Applying an orthographic projection of these dual tetrahedra along the axis of the paraboloid yields a reciprocal planar form $\mathbf{F}(v, e, f)$

and force $\mathbf{F}'(v', e', f')$ pair, where the force diagram is an instance of a Maxwell 2D diagram with edges being perpendicular to the corresponding ones in the form diagram.

3.1.2. Perspective projections of reciprocal polyhedra induced by a sphere

The Maxwell 2D diagrams $(\mathbf{F}, \mathbf{F}')$ can be also obtained through a polarity induced by a sphere as described in Section 2.6.1. In this case (Fig. 10), given the initial tetrahedral stress function, a polar reciprocal tetrahedron is first obtained through the polar transformation and then projected on the plane by means of a perspective projection from the centre of the sphere. It should be noticed that the orthographic projection in the previous paragraph can be also regarded as a special case of a perspective projection from the centre of the paraboloid of revolution. In the projective three-space, the centre of a quadric is defined as the pole of the plane at infinity according to that quadric. Since the paraboloid is tangent to the plane at infinity, its centre is self-conjugate to the plane at infinity and thus lies at infinity as well. As result, in the Euclidean embedding, this perspective projection from infinity manifests itself as an orthographic projection. As a result, with both methods the same Maxwell 2D reciprocal can be obtained, even though the polar polyhedron is different in the two cases. Furthermore, it can be observed that the polarity induced by a sphere is equivalent to the plane and origin construction mentioned above. In fact, in both cases, the projections of the reciprocal vertices are perspective projections from a point and the projection ray is perpendicular to the polar planes, implying that the point can be conceived as the centre of a sphere.

3.2. Cremona 2D

3.2.1. Orthographic projections of reciprocal polyhedra induced by a null-polarity

Given a self-stressed 2D form diagram (\mathbf{F}) with an underlying planar graph, a Cremona 2D force reciprocal (\mathbf{F}'), with reciprocal edges parallel to the corresponding edges of the form diagram, can be obtained through the previously described null-polarity. The force diagram is then the projection of the dual polyhedron (\mathbf{P}') reciprocal to the polyhedral Airy stress function (\mathbf{P}) of the form diagram, in the direction along the central axis a and on a plane π orthogonal to the central axis itself (*orthographic plane*) (Fig. 11). We observe that the null-polarity essentially produces the same pair of reciprocal form and force diagrams as in the Maxwell 2D cases described above with the only difference that one of the diagrams is rotated by 90° . Thus, corresponding reciprocal edges are now parallel.

3.3. Rankine 3D

3.3.1. Projections of reciprocal 4-polytopes induced by a hyper-quadric

It is possible to extend the above constructions for quadrics to spatial trusses, which are projections of simply connected 4D stress functions $\mathbf{P}(v, e, f, c)$, $\mathbf{P}'(v', e', f', c')$ (Maxwell Rankine stress functions). Essentially, the constructions are exactly the same, just one dimension up. Topologically, this type of structures comprise internal cells enclosed by an external boundary and in three-space they look like cellular or polyhedral structures. In this case, vertices v of a 4-polytope map to hyperplanes c' which intersect to produce reciprocal force cells (in the same fashion as for 2D trusses where the polar planes of the polyhedral Airy stress function intersect and to faces). Thus, creating a reciprocal 4-polytope. The 3D projections of this higher-dimensional pair will result in a pair of spatial reciprocal form and force diagrams $\mathbf{F}(v, e, f, c)$, $\mathbf{F}'(v', e', f', c')$ where form edges (e) correspond to reciprocal perpendicular force faces (f') (Rankine 3D reciprocals). For the simplest case of a simply connected 4-polytope $\mathbf{P}(5, 10, 10, 5)$ (5-simplex), a polarity (as described in Section 2.6.1) is used to map it to its reciprocal $\mathbf{P}'(5, 10, 10, 5)$ (Fig. 12). After projecting the 4-polytopes back to three-space, the three-dimensional form and force diagrams are obtained. Apart from these cases of spatial structures, this method can be also used for rotationally symmetric 1-cell structures such as tensegrities through the geometrical technique of coning (McRobie, 2016).

3.4. Cremona 3D

For a given 3D form diagram with underlying 2D planar graph and a state of self-stress, it is possible to derive a Cremona 3D reciprocal force diagram (Crapo, 1979), where spatial form edges correspond to reciprocal parallel force edges (Fig. 13). Contrary to the Cremona 2D case, however, these three-dimensional reciprocal diagrams are not projections of 4D reciprocal stress functions through a null-polarity, which as already mentioned, is not defined mathematically in even dimensions. Cremona 3D reciprocals have been investigated in the work of Sauer (1970) and more recently in Micheletti (2008); Wallner and Pottmann (2008), and Tachi (2012).

More generally, since the 2D graphs of 3D form and force diagrams are usually not planar, edge-to-edge reciprocity between the 3D diagrams is not possible in such cases (Jasienski et al., 2016). Different solutions to derive reciprocal form and force diagrams with corresponding edges parallel and with underlying non-planar graphs can be found in literature for both 2D and 3D cases. For 2D trusses, the method suggested by Bow consists in the addition of an extra node at the intersection of bars (Bow, 1873). In the

case of 3D trusses, Maxwell observed that the static equilibrium of a spatial structure can be evaluated using 2D graphic statics by studying its three orthogonal two-dimensional projections and their corresponding Airy stress functions. More recently, Micheletti proposed a matrix-based procedure for the construction of point-symmetric reciprocal force diagrams for some 3D self-stressed networks (Micheletti, 2008).

Given a generic 3D form diagram in static equilibrium, various possible configurations of 3D force diagrams with corresponding edges parallel can be generated and transformed using the vector-based approach to 3D graphic statics (D'Acunzio et al., 2017). Unlike Rankine 3D diagrams, vector-based 3D diagrams do not obey a 4D duality between their geometrical elements. In particular, they follow a rather 3D duality in the sense that form edges (e) map to force edges (e') and form vertices (v) to force faces (f'), however; they generally have duplicate edges (Fig. 14). As a result, vector-based diagrams are in general not reciprocal.

4. Implementation and results

Following the constructions described above, we present several case studies. For the two-dimensional case, we start with the Maxwell 2D reciprocals and the corresponding Airy stress functions for two of the figures found in Maxwell (1864) (Fig. 15). We then show the reciprocal polyhedral Airy stress function and the 2D projections in terms of a pair of form and force diagrams of a Pratt truss under external loading, and the equivalent self-stressed truss (Fig. 16). For the same case, we then apply polar transformations for the analysis and design of the geometry and we highlight, how this framework can incorporate several transformations, and their combinations, (Fig. 17) such as: local transformations induced by moving individual nodes; global projective transformations (scaling, shearing, rotating, etc.); as well as local and global transformations of the polyhedral Airy stress function. The latter can be particularly useful for cases where the external boundaries of a structure should remain fixed and the designer wants to change only the geometry of the internal structural members. Since in Maxwell's approach reciprocal form and force diagrams can interchange roles, it is important to notice that the transformations can equivalently happen in the form, or force diagram, or their corresponding stress functions in which case the other three reciprocal objects are directly updated. As a result, the designer can directly design and control the force diagram by changing the location of its vertices (which corresponds to altering lengths for the 2D case and face surface areas for the 3D case) and visually inspect and interact with the updated form. Moreover, after transforming the force diagram or any of the polyhedral stress functions, the result will be a new geometry under static equilibrium. In the case of local transformations and when the geometrical objects under transformation are not triangulated, the constraint should be imposed that the faces of the polyhedral stress functions remain planar. Lastly, as a known result in rigidity theory (Whiteley, 1982), convex Airy stress functions project to two-dimensional *spiderweb* structures: structures which are tension-only enclosed by a compressive hoop or equivalently structures which are compression-only enclosed by a tensile hoop. As a result, by manipulating the polyhedral Airy stress function it is possible to produce tension-only/compression-only, or tension and compression structures. Furthermore, we should highlight how any boundary conditions/ external forces can be incorporated in the initial geometry, which can be subsequently seen and analysed as an equivalent self-stressed truss (Fig. 18). Any external concurrent forces in equilibrium can be applied either on the structural boundary or on internal nodes of the structure and they result in a further subdivided equivalent self-stressed truss. In the general case of non-concurrent external forces in equilibrium a funicular

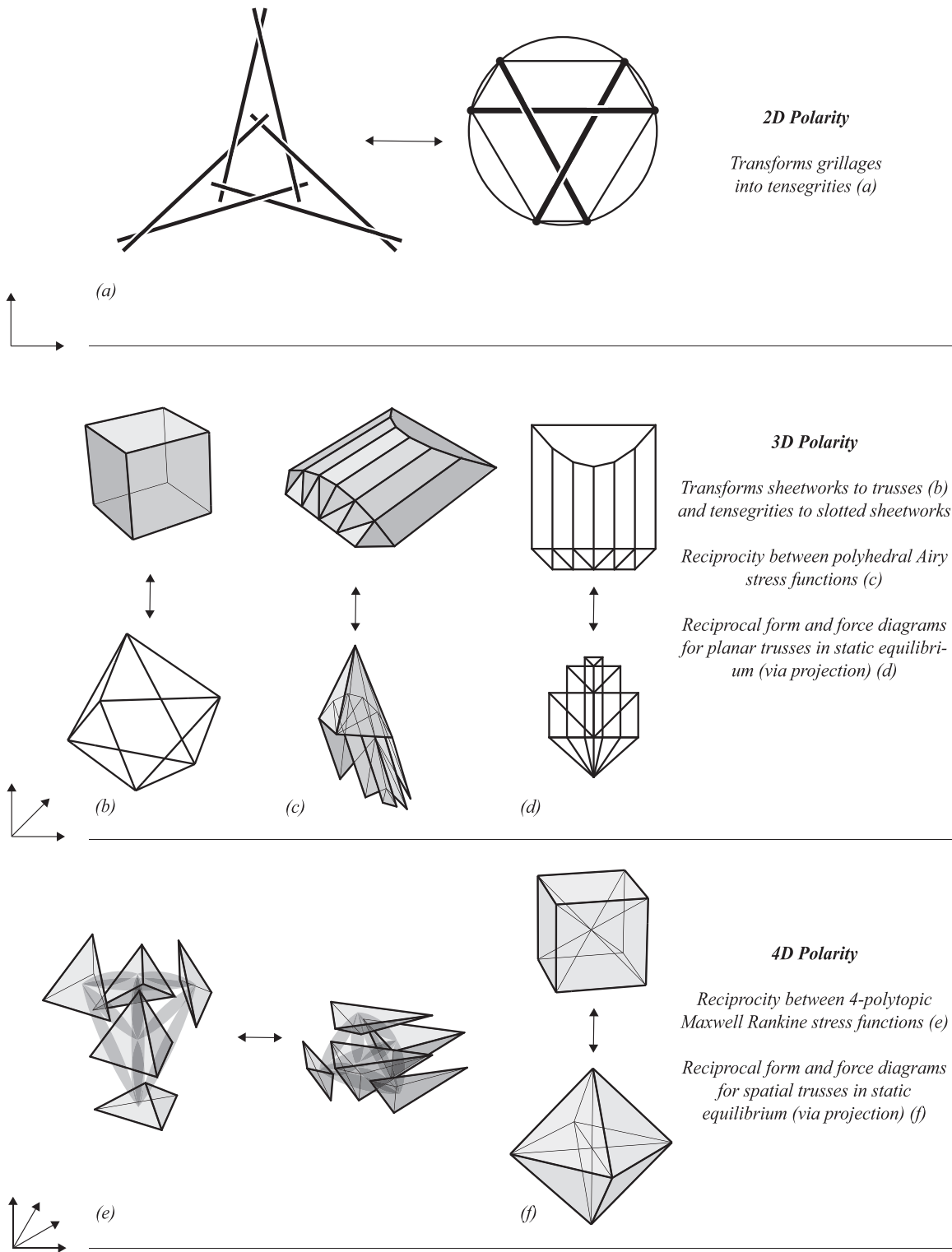


Fig. 23. The application of polarities in structural analysis and design.

structure can be also added to produce a diagram which is a projection of a simply-connected stress function as explained in detail in (McRobie et al., 2016) (Fig. 16).

For the spatial case, we apply the constructions to trusses which are projections of simply connected 4-polytopes, such as

the spoked cube, and rotationally symmetric tensegrity structures, the Jessen icosahedron and the 3-prism for which we obtain their Rankine 3D reciprocals (Fig. 19) through coning (adding an internal node and thus creating several internal cells in these otherwise 1-cell structures). Furthermore, similar to the Pratt truss we

apply this framework to the design and analysis of spatial trusses allowing for local and global transformations at the same time of the form of force geometries. A polyhedral spatial truss with three internal nodes (Fig. 20), and thus three states of self-stress, can undergo local transformations by moving external or internal nodes. Moreover, global projective transformations can be applied on its form diagram, force diagram, or corresponding stress functions, and can be combined with local transformations. As in the 2D case, these transformations always result in a new spatial geometry in static equilibrium.

5. Further applications of polarities to structural engineering theory

In the context of structural engineering theory, polar transformations have not only been used in relation to graphic statics. Tarnai (1989) used polarities in the projective two-space to establish transformations between planar tensegrity structures and grillages (*reciprocal frames*) (Fig. 21), in a projective geometry framework that guarantees duality. These two types of structures obey a 2D duality principle with the number of joints and bars of the tensegrity corresponding to the number of beams and joints of the grillage. As observed by Rankine (1864) and Whiteley (1987) the static and kinematic properties of a structure, including its stiffness, are preserved under projective transformations. Moreover, Tarnai (1989) proved algebraically that the rank of the equilibrium matrix is preserved under a polarity which maps tensegrities to grillages and vice versa. As a result, these types of structures can be transformed into each other while maintaining their basic properties such as infinitesimal rigidity (Tarnai, 1989).

In the projective three-space, polarities can be applied for the construction of lattice (truss) and dual plate structures (*hinged sheetworks*), where plates rigid in their planes are connected through shear-resistant edges. This has been studied in detail by Wester (1989, 2011) as well as by rigidity theorists (Whiteley, 1987). Following this 3D dualism (Fig. 1 middle column), truss nodes map to plate faces, truss bars to shear-resistant edges, and consequently axial forces to shear forces. The two structures obey Euler's polyhedral formula, have the same Gaussian curvature (Wester, 1989), static and infinitesimal behaviour, rigidity properties, and degrees of freedom (Whiteley, 1987). The construction proposed by Wester for deriving dual lattice and plate structures is essentially a polarity induced by a sphere (Fig. 22). In this way, the axial loads in the lattice structure regarded as force vectors are transformed into moment vectors that produce shear forces in the edges of the plate structure. The shear forces relate to the axial forces by a ratio r_f which is calculated as $r_f = l/\sin(a)$, where l is the length of the reciprocal truss bar and a is the angle between the position vectors of the nodes of the bar regarding the centre of the sphere as the origin O (Wester, 1989). As a result, the structural and geometrical analysis and design of a plate structure can be performed on the reciprocal truss and then mapped back to the original structure and vice versa (Wester, 2011) – potentially simplifying the structural problem, when the reciprocal structure is easier to solve.

The work of Tarnai and Wester meets and can be generalized through rigidity theory. Whiteley (1987), extended spatial trusses to become spatial tensegrities and discusses how they can be transformed under a polarity to special cases of hinged sheetworks, (*slotted sheetworks*) in which plates meeting at an edge can not only rotate but also slide with respect to each other on one direction along the slotted hinge. This result, even though not commonly used, generalises Tarnai's tensegrity – grillage polar correspondence in two-dimensions to its three-dimensional analogue. Thus, for the same structure the dimension of the polar transformation can give different insights; for a spatial tensegrity a 4D po-

larity can give a reciprocal Rankine force diagram and a 3D polarity can transform the tensegrity to a completely different type of structure which is also in static equilibrium. Consequently, polarities can provide a very elegant, useful and unified framework for applications in design, analysis, and structural transformations.

Discussion & conclusions

We discussed and visualised Maxwell's reciprocal constructions via polar transformations based on higher dimensional stress functions. We explained these constructions through the scientific frame of the second half of 19th century and placed them within a projective geometry framework. We thus discussed the underlying fundamental geometry of contemporary graphic statics. Moreover, we generalised these polar transformation methods following a matrix analysis approach to obtain a method for generating any type of reciprocals for n -dimensional trusses in static equilibrium when these are projections of simply connected $(n+1)$ -polytopes. We discussed how other already existing research and applications in structural design and analysis are underpinned by exactly the same geometrical principles. As a result, the abstract geometrical framework of projective geometry and polarities can have different physical manifestations and engineering applications (Fig. 23): using a 2D polarity it is possible to transform grillages into tensegrities (Tarnai, 1989); using a 3D polarity plate structures can be converted into trusses (Wester, 1989) and tensegrities to slotted sheetworks (Whiteley, 1987); furthermore, 3D polarity can be used to derive reciprocal polyhedral Airy stress functions and force diagrams of planar trusses. By means of a 4D polarity, it is possible to derive reciprocal Maxwell-Rankine stress functions and Rankine 3D reciprocals for spatial trusses.

This proposed framework based on polarity and projective geometry, relies on direct mathematical operations and allows for the geometrical study, design, analysis, and form finding of planar and spatial structures in a unified way. Thanks to the use of contemporary computer aided design tools, which allow for an intuitive and visual definition of geometrical objects, the proposed theoretical framework is implemented into a series of computational routines. As a result, the theoretical framework is translated into a computational framework which allows for a real-time and smooth application of polar transformations to several structural design and analysis tasks.

Acknowledgements

The authors are grateful to Prof. C. R. Calladine for fruitful conversations and for pointing out historical references with regards to Maxwell and his contemporaries, and to Bill Baker for his insightful comments and suggestions. MK is a recipient of an EPSRC studentship through the University of Cambridge. PD is a recipient of a studentship from ETH Zürich. FAM receives a salary from the University of Cambridge. The authors thankfully acknowledge the EPSRC for funding this research through the EPSRC Centre for Doctoral Training in Future Infrastructure and Built Environment (EPSRC grant reference number EP/L016095/1).

Appendix A

When the pole P is inside the conic γ on the projective plane, the polar line p can be found as follows (Fig. 1): from P take any two arbitrary lines p_1, p_2 which will intersect γ in points A, B and C, D respectively; from these two pairs of points take the corresponding pairs of tangent lines a, b and c, d with regards to γ ; these two pairs of lines will intersect in points AB and CD . These two points define the polar line p . Equivalently, in projective three-space for a pole P inside a quadric Γ the polar plane π can be

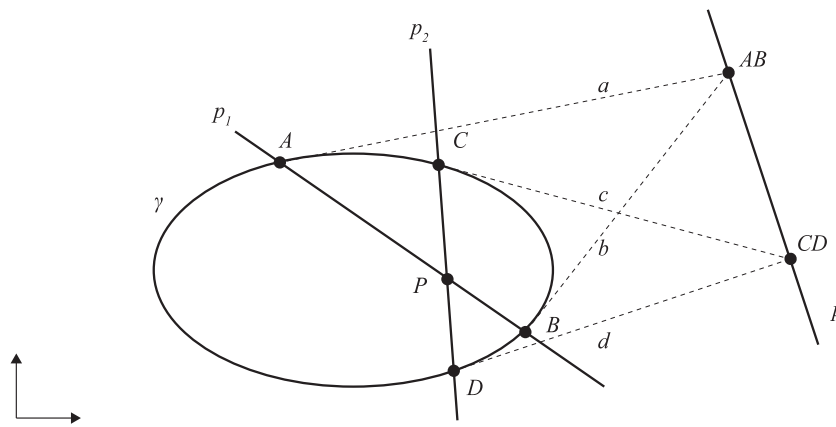


Fig. A.1. Polarity on the plane between a point (P) and a line (p) induced by a conic in 2D projective geometry for the case of P being internal with regards to the conic.

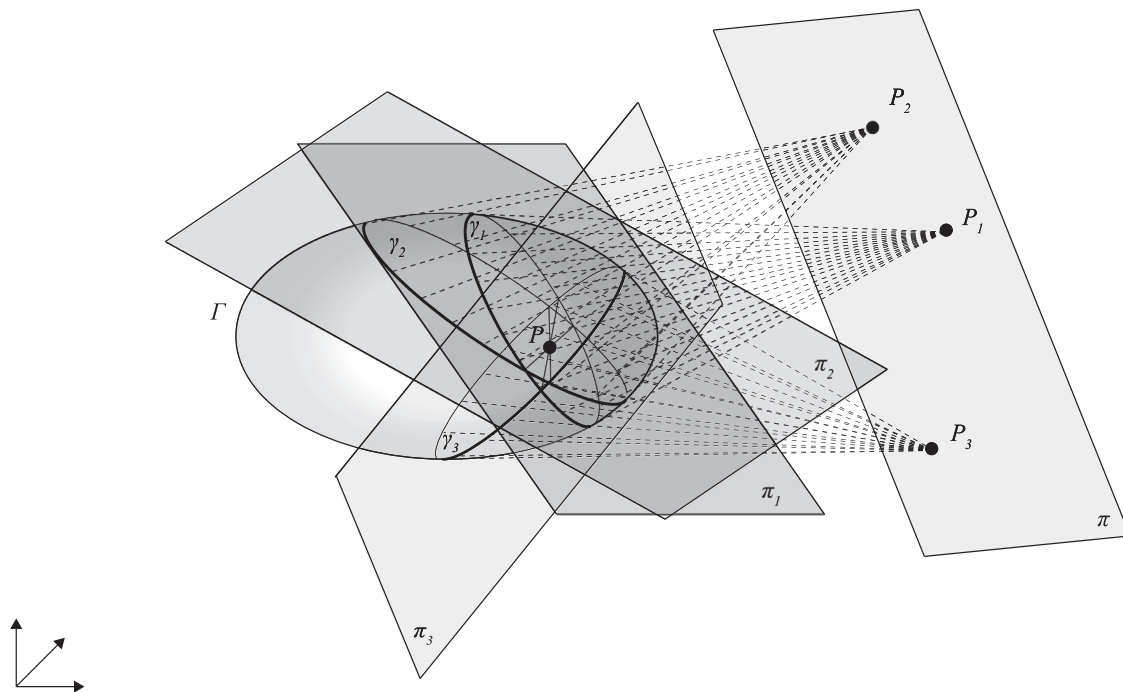


Fig. A.2. Polarity in space between a point (P) and a plane (π) induced by a quadric in 3D projective geometry for the case of P being internal with regards to the quadric.

found as follows (Fig. 2): from P take any three arbitrary planes π_1, π_2, π_3 , which intersect Γ in the planar curves $\gamma_1, \gamma_2, \gamma_3$ respectively; from these three curves take the corresponding tangent cones which define points P_1, P_2, P_3 . These three points define the polar plane π .

References

- Airy, G.B., 1862. On the strains in the interior of beams. *Philos. Trans. R. Soc. Lond.* 153, 49–79.
- Akbarzadeh, M., Van Mele, T., Block, P., 2015. On the equilibrium of funicular polyhedral frames and convex polyhedral force diagrams. *Comput.-Aided Des.* 63, 118–128.
- Baer, R., 1945. Null systems in projective space. *Bull. Am. Math. Soc.* 51, 903–906.
- Beghini, L.L., Carrion, J., Beghini, A., Mazurek, A., Baker, W.F., 2013. Structural optimization using graphic statics. *Struct. Multidisc. Optim.* 49, 351–366.
- Block, P., Ochsendorf, J., 2007. Thrust network analysis: a new methodology for three-dimensional equilibrium. *J. Int. Assoc. Shell Spatial Struct.* 48, 167–173.
- Bow, R.H., 1873. *Economics of Construction in Relation to Framed Structures*. E. & F. N. Spon, London.
- Charlton, T.M., 1982. *A History of Theory of Structures in the Nineteenth Century*. Cambridge University Press, Cambridge.
- Chasles, M., 1830. Note sur les propriétés générales du système de deux corps semblables entr'eux. *Bulletin des Sci. Mathématiques, Astron. Phys. et Chem.* (in French) 14, 321–326.
- Chasles, M., 1875. *Aperçu Historique Sur L'origine Et Le Développement Des Méthodes En géométrie, Particulièrement De Celles Qui Se Rapportent à La Géométrie Moderne*. Gauthier-Villars, Paris.
- Coxeter, H.S.M., 1974. *Projective Geometry*, Second. ed Springer-Verlag, New York.
- Coxeter, H.S.M., 1969. *Introduction to Geometry*, Second. ed John Wiley & Sons, INC, New York.
- Crapo, H., 1979. Structural rigidity. *Struct. Topol.* 1, 26–45.
- Crapo, H., Whiteley, W., 1994. 3-stresses in 3-space and projections of polyhedral 3-surfaces: reciprocals, liftings and parallel configurations. Preprint.
- Crapo, H., Whiteley, W., 1993. Plane self-stresses and projected polyhedra I: The basic pattern. *Struct. Topol.* 20, 55–78.
- Cremona, L., 1890. *Graphical Statics, Two Treatises on the Graphical Calculus and Reciprocal Figures in Graphical Statics*. Clarendon Press, Oxford.
- Cremona, L., 1872. *Le Figure Reciproche Nella Statica Grafica, 1879th ed Tipografia di Giuseppe Bernardoni, Milano*.
- Cremona, L., 1873. *Elementi Di Geometria Proiettiva As Uso Degli Istituti Tecnici*. G. B. Paravia E Comp., Torino.
- Culmann, K., 1866. *Die Graphische Statik*. Meyer und Zeller.
- D'Acunto, P., Ohlbrock, P.O., Jasienski, J.P., Fivet, C., 2016. Vector-based 3d graphic statics (part i): evaluation of global equilibrium. In: *Proceedings of the IASS 2016*.
- D'Acunto, P., Jasienski, J.P., Ohlbrock, P.O., Fivet, C., 2017. Vector-based 3d graphic statics: transformations of force diagrams. In: *Proceedings of the IASS 2017*.
- Fivet, C., 2016. Projective transformations of structural equilibrium. *Int. J. Space Struct.* 31, 135–146. doi:10.1177/0266351116660796.

- Fraternali, F., Carpentieri, G., 2014. On the correspondence between 2D force networks and polyhedral stress functions. *Int. J. Space Struct.* 29, 145–159.
- Gray, J., 2010. *Worlds Out of Nothing*. SUMS. Springer, New York.
- Harman, P.M., 2004. Maxwell, James Clerk (1831–1879), Physicist. *Oxford Dictionary of National Biography*.
- Harman, P.M., 1998. *The Natural Philosophy of James Clerk Maxwell*. Cambridge University Press, Cambridge.
- Harman, P.M., 1990. *The Scientific Letters and Papers of James Clerk Maxwell*. Cambridge University Press, ed, Cambridge.
- Heyman, J., 1995. *The Stone Skeleton*. Cambridge University Press, Cambridge.
- Jasienski, J.P., D'Acunto, P., Ohlbrock, P.O., Fivet, C., 2016. Vector-based 3d graphic statics (part ii): construction of force diagrams. In: *Proceedings of the IASS 2016*.
- Konstantatou, M., McRobie, A., 2016. Reciprocal constructions using conic sections & Poncelet duality. In: *Proceedings of the IASS 2016*.
- Kurrer, K.-E., 2008. *The History of the Theory of Structures, from Arch Analysis to Computational Mechanics*. Ernst & Sohn, Germany.
- Lee J., Mele T.V., Block P., 2018. Disjointed Force Polyhedra. To appear in: *Computer-Aided Design*, doi:10.1016/j.cad.2018.02.004
- Maxwell, J.C., 1870. On reciprocal figures, frames, and diagrams of forces. *Trans. R. Soc. Edinburgh* 26, 1–40.
- Maxwell, J.C., 1867. On the application of the theory of reciprocal polar figures to the construction of diagrams of forces. *Eng.* 24, 402.
- Maxwell, J.C., 1864. On reciprocal figures and diagrams of forces. *Lond Edinburgh Dublin Philos. Mag. J. Sci.* 27, 250–261.
- Mazurek, A., Beghini, A., Carrion, J., Baker, W.F., 2016. Minimum weight layouts of spanning structures obtained using graphic statics. *Int. J. Space Struct.* 31, 112–120.
- McRobie, A., 2017. The geometry of structural equilibrium. *R. Soc. Open Sci.* 4, 160759. doi:10.1098/rsos.
- McRobie, A., 2016. Maxwell and Rankine reciprocal diagrams via Minkowski sums for two-dimensional and three-dimensional trusses under load. *Int. J. Space Struct.* 31, 203–216.
- McRobie, A., Baker, W., Mitchell, T., Konstantatou, M., 2016. Mechanisms and states of self-stress of planar trusses using graphic statics, part II: Applications and extensions. *Int. J. Space Struct.* 31, 102–111.
- Micheletti, A., 2008. On generalized reciprocal diagrams for self-stressed networks. *Int. J. Space Struct.* 23, 153–166.
- Mitchell, T., Baker, W., McRobie, A., Mazurek, A., 2016. Mechanisms and states of self-stress of planar trusses using graphic statics, part I: the fundamental theorem of linear algebra and the airy stress function. *Int. J. Space Struct.* 31, 85–101.
- Möbius, A.F., 1833. Über eine besondere art dualer verhältnisse zwischen figuren im raume. In: *Crelle's J. für die reine und angewandte Mathematik*, pp. 317–341.
- Monge, G., 1794. *Geometrie Descriptive*. Baudouin, Paris.
- Muttoni, A., 2011. *The Art of Structures: Introduction to the Functioning of Structures in Architecture*. EPFL Press.
- Muttoni, A., Schwartz, J., Thürlimann, B., 1996. *Design of Concrete Structures With Stress Fields*. Birkhäuser.
- Ohlbrock, P.O., D'Acunto, P., Jasienski, J.P., Fivet, C., 2016. Vector-based 3d graphic statics (part iii): designing with combinatorial equilibrium modelling. In: *Proceedings of the IASS 2016*. Tokyo.
- Poncelet, J.V., 1822. *Traité Des Propriétés Projectives Des Figures*. Bachelier, Paris.
- Rankine, M., 1864. Principle of the equilibrium of polyhedral frames. *Lond. Edinburgh Dublin Philos. Mag. J. Sci.* 27, 92.
- Rankine, W.J.M., 1858. *A Manual of Applied Mechanics*. Richard Griffin and Company, ed. Glasgow.
- Rosenfeld, B.A., 1988. *A History of Non-Euclidean Geometry, Evolution of the Concept of a Geometric Space, Studies in the History of Mathematics and Physical Sciences*. Springer-Verlag, New York.
- Sauer, R., 1970. *Differenzgeometrie*. Springer, Berlin.
- Smith, C., 1886. *Elementary Treatise On Solid Geometry*. Macmillan and Co., London.
- Tachi, T., 2012. Design of infinitesimally and finitely flexible origami based on reciprocal figures. *J. Geom. Graph.* 16, 223–234.
- Tarnai, T., 1989. Duality between plane trusses and grillages. *Int. J. Solids Struct.* 25, 1395–1409.
- Varignon, P., 1725. *Nouvelle Mecanique ou Statique*. Claude Jombert, Paris.
- Vaisman, I., 1997. *Analytical Geometry, Series on University Mathematics*. World Scientific Publishing Co. Pte. Ltd., Singapore.
- Wallner, J., Pottmann, H., 2008. Infinitesimally flexible meshes and discrete minimal surfaces. *Monat. Math.* 153, 347–365.
- Wester, T., 2011. 3-D form and force language proposal for a structural basis. *Int. J. Space Struct.* 26.
- Wester, T., 1989. The dualistic symmetry between plane- and point-based spatial structures. *Symmetry of Structure*.
- Whiteley, W., 1979. Realizability of polyhedra. *Struct. Topol.* 1, 46–58.
- Whiteley, W., 1982. Motions and stresses of projected polyhedra. *Struct. Topol.* 7, 13–38.
- Whiteley, W., 1987. Rigidity and polarity i: statics of sheet structures. *Geom. Dedicata* 22, 329–362.
- Zalewski, W., Allen, E., 1998. *Shaping Structures: STATICS*. John Wiley & Sons, INC., New York.
- Zalewski, W., Allen, E., 2009. *Form and Forces: Designing Efficient, Expressive Structures*. John Wiley & Sons, INC., New York.
- Zanni, G., Pennock, G.R., 2009. A unified approach to the static analysis of axially loaded structures. *Mech. Mach. Theory* 44, 2187–2203. *Sci. J. Int. Fed. Theory Mach. Mech.* Pergamon Press Ltd., Oxford.