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Vector-Based 3D Graphic Statics: Transformations of Force Diagrams

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Abstract

The reciprocity between form and force diagrams in 2D graphic statics makes it possible to manipulate the form diagram while directly evaluating the redistribution of the forces within the force diagram. Conversely, after modifying the force diagram, the consequent transformation of the form diagram can be assessed at once. In the case of vector-based 3D graphic statics, the reciprocity between the diagrams is generally not achieved. This paper describes a series of transformations that can be applied to a vector-based 3D force diagram while allowing the corresponding 3D form diagram to adjust accordingly. Two categories of manipulations of the force diagram are described: global transformations that affect simultaneously all the elements of the diagram and local transformations, which permit the manipulation of individual elements of the diagram. Thanks to these transformations, the adjustment of the magnitude and the direction of the forces in vector-based 3D force diagrams can be used as an active operation in the structural design process.

Keywords: vector-based 3D graphic statics, 3D force diagram, global transformations, local transformations, parallel transformations, projective transformations, constraint-driven transformations

1. Introduction

One of the most peculiar features of 2D graphic statics is the geometric reciprocity between form and force diagrams. This property has been investigated in depth by Maxwell [12], who discussed the case of reciprocal diagrams in which corresponding edges are perpendicular to each other. This has been followed by the findings of Cremona [6], who defined a geometric procedure to generate reciprocal diagrams with corresponding parallel edges. Thanks to the reciprocity of the diagrams, on the one hand, it is possible to manipulate the form diagram and evaluate directly the effects on the distribution of the forces in the force diagram. On the other hand, the magnitude and the direction of the forces can be modified in the force diagram while assessing the resulting transformation of the form diagram. As outlined by Huerta [9], the transformation of the diagrams has been used as a fundamental operation for structural design since the introduction of 2D graphic statics.

In relation to 3D graphic statics, global parallel transformations that preserve the static equilibrium of a given form diagram have been extensively reviewed by Fivet [8], who has also introduced a geometric procedure to deal with non-parallel projective transformations. With respect to polyhedron-based 3D graphic statics, a method has been described by Akbarzadeh *et al.* [1] to build the reciprocal force diagram of a given form diagram; recursive subdivision operations of the force diagram that preserve its reciprocity to the form diagram have been also investigated by the same authors. This approach has been recently extended by Lee *et al.* [11] with the definition of a series of additional transformations that are applicable to the polyhedron-based force diagram. As shown by Jasienski *et al.* [10], contrary to the polyhedron-based approach, the reciprocity between the diagrams is generally not achieved in vector-based 3D graphic statics.

The aim of this paper is to explore a series of transformations that can be applied to the vector-based 3D force diagram of a spatial network in equilibrium (such as a pin-jointed framework or a strut-and-tie model within a continuum of material) while adjusting the 3D form diagram accordingly. These transformations ensure the parallelism between corresponding edges in the form and force diagrams and keep their topology unchanged. Two categories of manipulations are described: global and local transformations. The possibility of manipulating the force diagram by means of geometric transformations is particularly relevant within the structural design process. Thanks to this opportunity, the designer is able to control directly the distribution of the forces within a given structure and steer its load-bearing behaviour towards a desired one. In this way, the adjustment of the magnitude and the direction of the forces in a vector-based 3D force diagram can be used as an active operation in the structural design process. This promotes an approach to design based on the control of the forces, which is complementary to the one focused on the form.

2. Vector-based 3D Graphic Statics

In vector-based 3D graphic statics, the force diagram of a spatial network in equilibrium is built out of vectors. As pointed out by Maxwell [12], this approach allows for the construction of a force diagram for any given form diagram in equilibrium and it never becomes geometrically impossible as long as the problem is mechanically possible. As in the case of 2D graphic statics, the 3D force diagram is the assembly of the closed force polygons representing the equilibrium of the forces acting on the nodes of the form diagram; as such, to each edge of the form diagram corresponds a pair of parallel and opposite force vectors in the force diagram. These two vectors are usually overlapped in a single edge. If all the pairs of vectors are overlapped, the resulting form and force diagrams are reciprocal. As proved by Whitney [22], this is possible only if the underlying graph of the form diagram is planar. Spatial structures with planar graphs are, for example, meshes composed of bars in space (Mitchell *et al.* [15]) and specific 3D configurations, including, among others, some classical tensegrity structures (Micheletti [14]) and the dependent octahedron (Crapo [5]). Their vector-based 3D reciprocal force diagrams have been denominated in literature *Cremona Reciprocals* (Crapo [5]). However, since 3D form and force diagrams are usually not planar, edge-to-edge reciprocity between the 3D diagrams is generally not possible (Jasienski *et al* [10]). Various approaches to extend the concept of reciprocity to form and force diagrams with underlying non-planar graphs have been proposed for both 2D and 3D cases. Among these, the method of Bow [3] of adding an extra node at the intersection of bars in 2D pin-jointed frameworks, the proposal of Crapo and Whiteley [4] of reciprocals as infinite frameworks and the approach of Micheletti [14] for the reciprocal diagrams of some 3D self-stressed networks.

In order to construct the 3D force diagram $\mathbf{F}^*$ of a given 3D form diagram $\mathbf{F}$ in equilibrium with an underlying planar topological diagram (graph) $\mathbf{T}$, for each vertex V_i of $\mathbf{T}$, the connected edges e_i can be listed following a defined reading order cycle (Bow [3]). If external forces are present, these are connected to one node in the topological diagram and regarded as edges (Jasienski *et al.* [10]). The distribution of the inner forces in the structure is then assessed node-by-node and the nodal 3D force polygons are created; the lists of edges are used to order the vectors in these force polygons. If the structure is externally loaded, a closed force polygon of the external forces, representing the external global equilibrium (D'Acunto *et al.* [7]), is also generated. Following this approach, a reciprocal 3D force diagram $\mathbf{F}^*$ can be eventually assembled out of the individual nodal 3D force polygons of $\mathbf{F}$.

In case the underlying topological diagram $\mathbf{T}$ of the given 3D form diagram $\mathbf{F}$ is not planar (Figure 1), a possible strategy is here proposed to modify $\mathbf{T}$ and make it planar ($\mathbf{T_P}$), without altering the overall static equilibrium of the structure. The vertices V_i of $\mathbf{T}$ are first repositioned in order to minimize the amount of intersecting edges. Starting with the first intersection, one of the intersecting edges (e_{st}) is replaced by two new edges (e_s and e_t) that are respectively connected to the vertices V_s and V_t of e_{st} and to the node of the external forces (Figure 1). Concurrently, in the 3D form diagram $\mathbf{F}$, the edge $\mathbf{f_{st}}$ corresponding to e_{st} is substituted by a pair of equal and opposite external forces ($\mathbf{f_s}$ and $\mathbf{f_t}$), each force being aligned to $\mathbf{f}_{st}$ and applied to one of its nodes (P_s and P_t). The procedure is repeated for every other intersection in **T**, until a planar graph $\mathbf{T_P}$ is obtained. The inner forces are then evaluated and the force vectors are arranged in the nodal 3D force polygons following a defined reading order cycle on the vertices of $\mathbf{T_P}$. The force polygons are eventually assembled into one unique 3D force diagram $\mathbf{F}^*$, where the external forces generate themselves a closed force polygon. For each pair of newly introduced external forces ($\mathbf{f}_s$ and $\mathbf{f}_t$), a pair of non-overlapping vectors ($\mathbf{f}_s^*$ and $\mathbf{f}_t^*$) is found in $\mathbf{F}^*$. This configuration of the 3D force diagram is comparable to the double-layer one described by Jasienski *et al.* [10], but it results generally in a more compact force diagram.

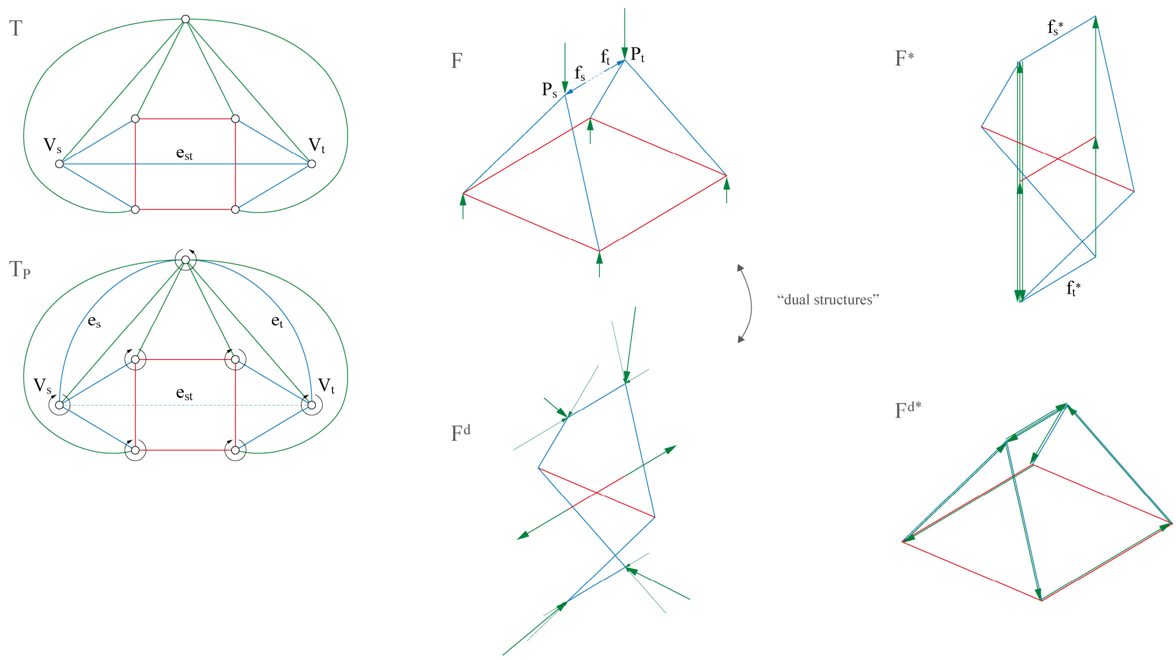

Figure 1: Topological diagram **T**, 3D form diagram **F** and 3D force diagram $\mathbf{F}^*$ (red-tension, blue-compression, green-external forces)

The proposed diagrams are in general not reciprocal because of the presence of non-overlapping pairs of vectors in $\mathbf{F}^*$. Nevertheless, important dual properties between the initial **F** and $\mathbf{F}^*$ are still preserved making them interdependent. In particular, each node of the initial **F** corresponds to a 3D closed polygon in $\mathbf{F}^*$. Conversely, despite the presence of non-overlapping pairs of vectors, if $\mathbf{F}^*$ is regarded as a structure in its own right ($\mathbf{F}^\mathbf{d}$), its nodal force polygons can be assembled to generate $\mathbf{F}^{\mathbf{d}*}$, which is equivalent to the initial **F**, deprived of its external forces. As shown in 2D (Mitchell *et al.* [15]), in $\mathbf{F}^*$ the vectors corresponding to the external forces of the initial **F** are first removed and a set of new external forces is then introduced to put $\mathbf{F}^\mathbf{d}$ in static equilibrium.

3. Global Transformations of the 3D Force Diagram

Following the classification introduced by Fivet [8], a series of global transformations of the 3D force diagram $\mathbf{F}^*$ is here presented that allow the modification of the inner forces in a given spatial network. These transformations are applicable to the force diagrams of spatial networks in equilibrium, no matter their static or kinematic determinacy (Pellegrino and Calladine [18]). Moreover, these transformations are valid in both cases of self-stressed and externally loaded structures. As it is shown in the following, it is possible to use these transformations regardless of the presence of non-overlapping pairs of vectors in $\mathbf{F}^*$, without losing its interdependency to the 3D form diagram **F** as explained in the previous section.

3.1. Global Parallel Transformations

Global parallel transformations in space are also known as affine transformations. Apart from Euclidean transformations (rotation, translation, reflection), affine transformations include, among others, spatial uniform scaling, non-uniform scaling and shear.

Affine transformations are characterized by three peculiar properties (Pottman *et al.* [20]): straight lines (planes) are transformed into straight lines (planes); parallel lines (planes) are mapped into parallel lines (planes); the ratio of the lengths of two line segments on parallel lines is invariant throughout the transformation. Based on the first property, an affine transformation converts $\mathbf{F}^*$ into a new vector-based diagram $\mathbf{F}^{*\prime}$; the same applies to **F**, which is transformed into **F'**. Thanks to the second property, corresponding edges in $\mathbf{F}^*$ and **F** stay parallel to each other, if the same affine transformation is performed on both. The third property, which is particularly relevant in the case $\mathbf{F}^*$ has non-overlapping pairs of vectors, ensures that these vectors stay parallel and have the same length after the transformation. As a result, any kind of affine transformation can be applied to a given 3D force diagram $\mathbf{F}^*$ while the corresponding 3D form diagram **F** undergoes the same transformation. In the following, the application of a series of global parallel transformations is exemplified on the 3D force diagram $\mathbf{F}^*$ (Figure 2.a) of a spatial network in equilibrium with given external forces. Each transformation is also represented analytically in matrix form, considering the transformation of a generic point in space P (x, y, z, 1) described using homogeneous coordinates (Figure 2).

A spatial uniform scaling (Figure 2.b) can be defined by setting a scale factor s_3 as a real number and an origin point O, which is unaffected during the transformation. The origin point can be chosen independently for $\mathbf{F}^*$ and **F**. As a special case among affine transformations, angles are preserved in a uniform scaling, so that lines stay parallel to themselves. Fixed a Cartesian coordinate system (**x**, **y**, **z**), the scale factor s_3 is equally applied to each coordinate axis. When a 3D force diagram $\mathbf{F}^*$ undergoes a uniform scaling, the magnitudes of all the forces within the structure are modified proportionally to s_3. It is immediate that a given 3D force diagram $\mathbf{F}^*$ relates to an infinite number of 3D form diagrams **F** that are scaled uniformly in space with chosen scale factors and origin points. As it is well known in 2D graphic statics (Huerta [9]), thanks to this property it is always possible to scale uniformly **F** to meet specific metric constraints (e.g. a given span), without modifying the forces in $\mathbf{F}^*$.

To define a non-uniform scaling in space, three scale factors (s_1, s_2, s_3) as real numbers are fixed so that at least one of them is different from the others. An origin point O, which is unaffected by the transformation, is also set. After fixing a Cartesian coordinate system (**x**, **y**, **z**), each scale factor is applied to one of the coordinate axes. In case one of the scale factors (s_1) is different from 1 and the other two are equal to 1, a scaling (or stretching) along one coordinate axis occurs (Figure 2.c). All the elements that lie on the plane Ω_{s1}, which contains O and is perpendicular to the coordinate axis along which the scaling occurs, are unaffected by the transformation. If one or more forces in $\mathbf{F}^*$ are parallel to Ω_{s1}, a stretching of $\mathbf{F}^*$ in the direction perpendicular to Ω_{s1} keeps the magnitudes and directions of those forces unchanged, but not the position of their lines of action. If the forces are perpendicular to Ω_{s1}, only their magnitudes change proportional to the scale factor s_1 and their lines of action are unaffected in both position and direction.

A shear transformation in space is defined by two shear factors (s_{V1}, s_{V2}) along two different coordinate axes and an origin plane Ω_{sV} parallel to those two coordinate axes. In Figure 2.d a shear transformation is applied to $\mathbf{F}^*$ (respectively **F**) along the **x**-axis only, with a shear factor s_{V1} and the **xz**-plane as origin plane Ω_{sV}. Every node P_i of $\mathbf{F}^*$ (respectively **F**) is shifted parallel to Ω_{sV}, with a translation vector $\mathbf{v_i} = \delta_i\, s_{V1}\, \mathbf{x}$ (where δ_i is the signed distance between P_i and Ω_{sV}). The forces in $\mathbf{F}^*$ that are parallel to Ω_{sV} keep their magnitudes and directions unchanged during the transformation.

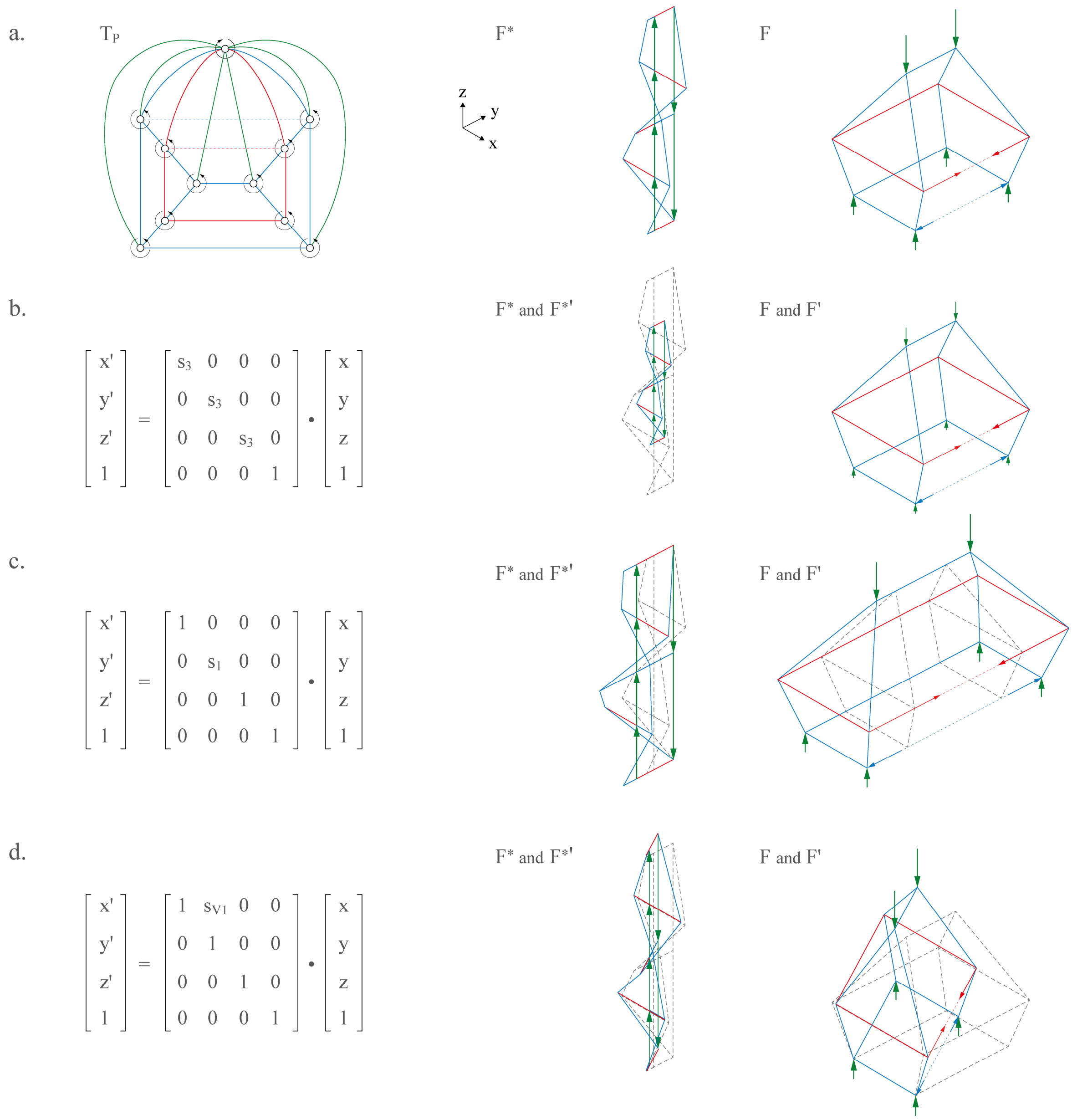

Figure 2: Global parallel transformations of a 3D force diagram (a. initial configuration; b. spatial uniform scaling; c. non-uniform scaling; d. shear transformation).

3.2. Semi-global Parallel Transformations

Under specific geometric or static conditions, it is possible to define transformations that only affect a subset of the 3D force diagram $\mathbf{F}^*$ while leaving the rest unchanged. In the following, two semi-global transformations are discussed in which the vectors of $\mathbf{F}^*$ are kept parallel to themselves.

In the case the 3D form diagram $\mathbf{F}$ of a given kinematic indeterminate structure has a symmetry plane Σ (Figure 3), it is possible to invert concurrently the loading state (tension-compression) of all those forces $\mathbf{f_{ij}}^*$ of $\mathbf{F}^*$ that correspond in $\mathbf{F}$ to edges $\mathbf{f_{ij}}$ perpendicular to Σ. All the other forces, external ones included, stay constant. In Figure 3, $\mathbf{F}^{*}$' is defined by inverting the loading state of the forces $\mathbf{f_{01}}^*/\mathbf{f_{10}}^*$, $\mathbf{f_{23}}^*/\mathbf{f_{32}}^*$, $\mathbf{f_4}^*$, $\mathbf{f_5}^*$, $\mathbf{f_6}^*$, $\mathbf{f_7}^*$, $\mathbf{f_{89}}^*/\mathbf{f_{98}}^*$ of $\mathbf{F}^*$. $\mathbf{F}$ is then modified accordingly into $\mathbf{F}$'.

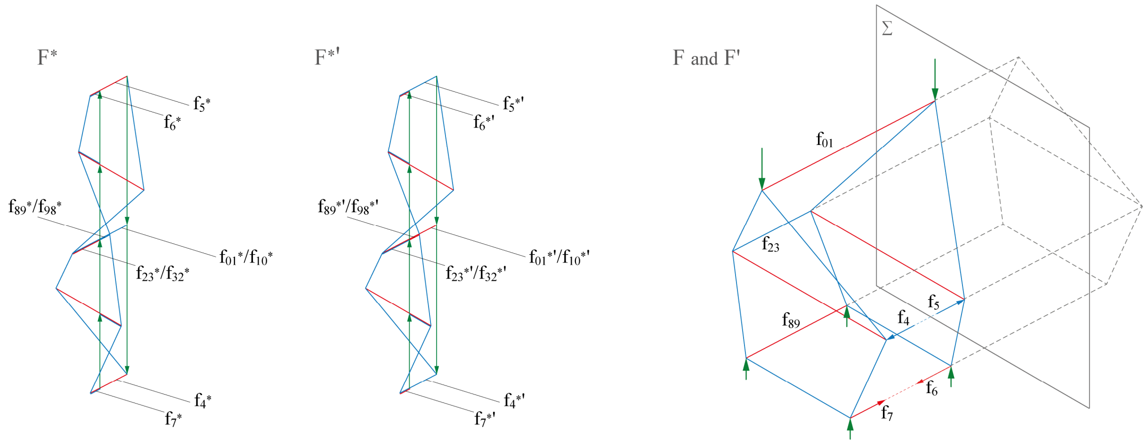

Figure 3: Inversion of the loading state in a 3D force diagram

In a statically indeterminate structure, to every degree of static indeterminacy, one infinity of inner forces distribution in static equilibrium can be found. As shown for 2D cases by Mitchell *et al.* [15] and by McRobie *et al.* [13], also in the 3D force diagram $\mathbf{F}^*$ of a statically indeterminate spatial network, degrees of freedom can be found that allow modifying a subset of the geometry of $\mathbf{F}^*$ while keeping the direction of all the forces $\mathbf{f_{ij}}^*$ unchanged (*consistent offsets*). These degrees of freedom can be used to transform $\mathbf{F}^*$ parametrically. Figure 4 shows an application of this principle to the 3D force diagram $\mathbf{F}^*$ of an externally loaded network with one internal degree of static indeterminacy. As the underlying topological diagram **T** of **F** is not planar, two non-overlapping pairs of vectors are present in $\mathbf{F}^*$. The external forces are kept constant and therefore the nodes P_0^*, P_1^*, P_2^* and P_3^* of $\mathbf{F}^*$ are fixed. The relative position x of the node P_x^* along the line of action of $\mathbf{f_{05}}^*/\mathbf{f_{50}}^*$ is regarded as a parameter. Given the position of $P_x^{*\prime}$, being the directions of all the forces in $\mathbf{F}^{*\prime}$ known, the positions of the remaining nodes of $\mathbf{F}^{*\prime}$ are unequivocally determined. In fact, these nodes can be found as the intersection of the lines of action of two forces or as the intersection of the line of action of one force and the plane passing through two forces. By varying x parametrically, the solution space of the inner forces distribution in the structure can be systematically explored while **F** stays constant.

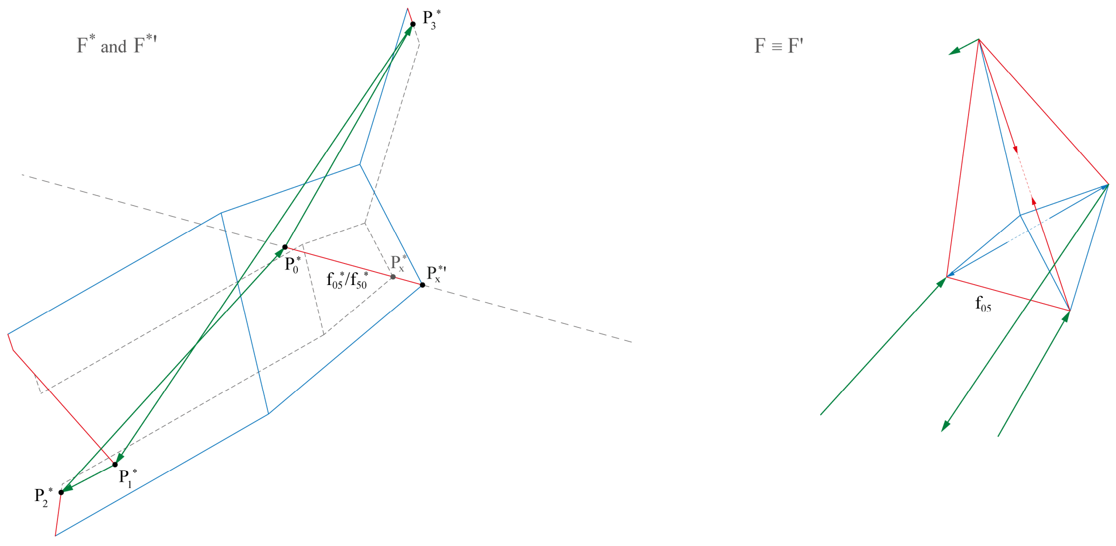

Figure 4: Transformation of the 3D force diagram of a network with one degree of static indeterminacy

3.3. Global Projective Transformations

Projective transformations are a generalization of affine transformations and map straight lines into straight lines, without maintaining lengths, angles, parallelism or ratio of lengths (Pottman *et al.* [20]). It is because these last two properties are not preserved that projective transformations are applicable only if a given 3D force diagram $\mathbf{F}^*$ has an underlining planar topological diagram **T**. In fact, if $\mathbf{F}^*$ has non-overlapping pairs of vectors, these should stay parallel and keep the same magnitude after the transformation, which would not be the case under a projective transformation. When possible, $\mathbf{F}^*$ can be transformed using the geometric construction introduced by Fivet [8]. As explained in Section 2, the modified 3D form diagram **F'** can be built by assembling the force polygons of $\mathbf{F}^{*\prime}$ regarded itself as a structure. Because of the aforementioned restriction on the planarity of **T** and the shortcomings described by Fivet [8] on the control of the external forces and support locations, the use of global projective transformations in the structural design process has very limited applications.

4. Local Transformations of the 3D Force Diagram

Contrary to global and semi-global transformations, which are applied at once to all the elements of the 3D force diagram or to a subset of them, local transformations allow for the manipulation of individual elements of the diagram. Thanks to these transformations, it is therefore possible to adjust the magnitude and direction of specific forces in $\mathbf{F}^*$ and assess the corresponding transformation of **F**.

4.1. Transformations of Kinematically Determinate Force Diagrams

In the particular case that a given 3D form diagram **F** has a 3D force diagram $\mathbf{F}^*$ that is itself a kinematically determinate network, any kind of transformation can be performed on $\mathbf{F}^*$ that modifies the position of its nodes without modifying its topology (Figure 5). Specifically, each node of $\mathbf{F}^*$ can be moved independently from the other nodes in any direction in space using a translation vector **t**. Moreover, any edge or face of $\mathbf{F}^*$ can be individually moved or scaled at will, by operating at the same time on its corresponding nodes. If external forces are applied to the structure, these can be modified directly by operating on the polygon of the external forces in $\mathbf{F}^*$. After $\mathbf{F}^*$ has been transformed into $\mathbf{F}^{*\prime}$, **F'** can be assembled as discussed in Section 2.

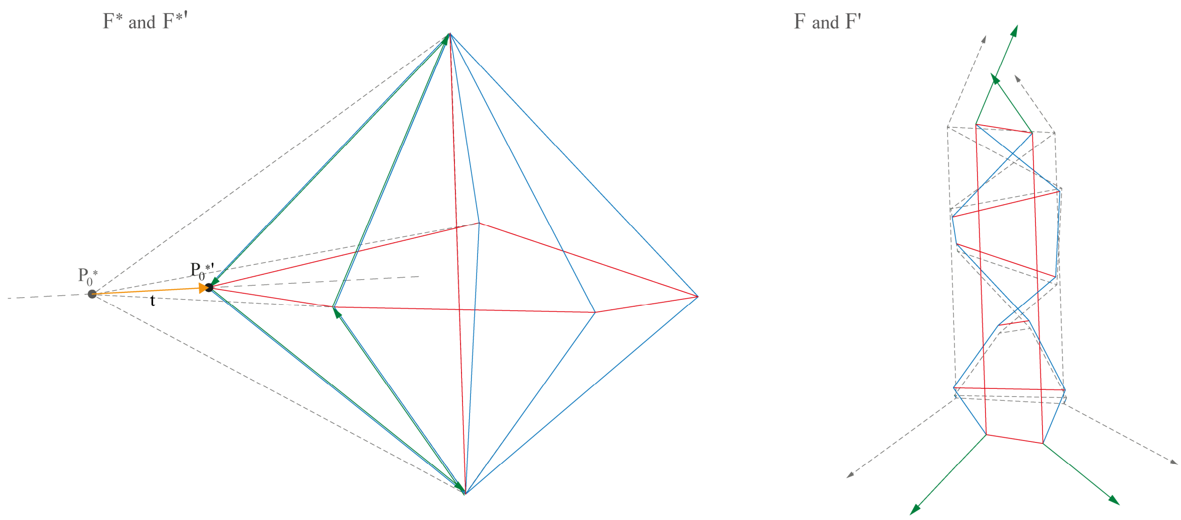

Figure 5: Local transformation of a kinematically determinate 3D force diagram

4.2. Transformations using the Combinatorial Equilibrium Modelling

Combinatorial Equilibrium Modelling (CEM) is an approach to structural design that is based on 3D graphic statics and graph theory (Ohlbrock *et al.* [17]). Given a spatial network in equilibrium, an initial 3D form diagram **F** and a force diagram $\mathbf{F}^*$ can be defined. If the members of the structure can be organized into trail members $\mathbf{t_{ij}}$ and deviation members $\mathbf{d_{ij}}$ (Ohlbrock *et al.* [17]), the required member lengths λ_{ij} of the trail members and force magnitudes μ_{ij} of the deviation members can be derived from the initial **F** and $\mathbf{F}^*$ respectively. Based on this setup, $\mathbf{F}^*$ can be then modified by changing directly the loading state (tension-compression) of $\mathbf{t_{ij}}^*$ and $\mathbf{d_{ij}}^*$ other than the values μ_{ij} of $\mathbf{d_{ij}}^*$ (Figure 6). Thanks to the CEM approach, it is possible to control individually the magnitudes and directions of the external loads applied to the structure. However, in order to meet specific constraints such as the location of the supports in **F**, an optimization process is generally required (Ohlbrock *et al.* [16]). In the example in Figure 6, the magnitudes of the forces $\mathbf{d_{01}}^*$ and $\mathbf{d_{23}}^*$ have been halved and the ones of $\mathbf{d_{56}}^*$, $\mathbf{d_7}^*$ and $\mathbf{d_8}^*$ have been doubled.

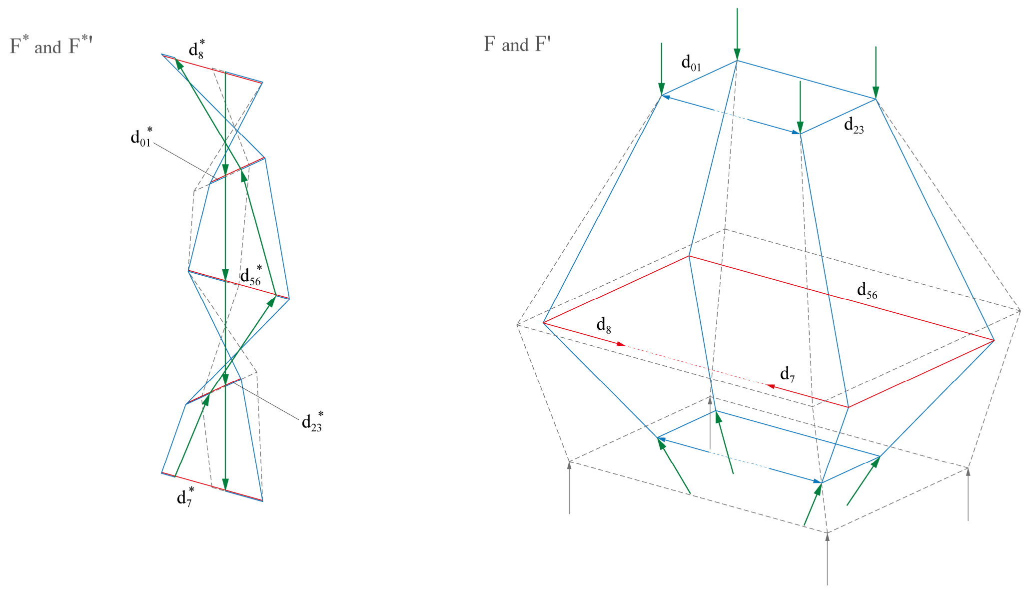

Figure 6: Local transformation of a 3D force diagram using CEM

4.3. Geometric Constraints and Numerical Methods

Numerical methods can be effectively used for the solution of specific structural problems, especially in the case of form finding. Significant examples of numerical approaches adopted in this field are the force density method (Schek [21]) and dynamic relaxation (Barnes [2]). The employment of numerical methods is particularly relevant for the solution of constraint-driven non-linear problems that cannot be solved otherwise using only direct geometric constructions. In comparison to the latter, numerical methods are based on iterative numerical approximations; as such, contrary to the transformations described in the previous sections, they normally require computational tools to be performed.

The transformation here exemplified is executed within the digital CAD environment *McNeel Rhino*, using the plug-in *Kangaroo2* by Piker [19]. The solver within the plugin is built around a specific implementation of dynamic relaxation, which is defined by its author as projection-based dynamic relaxation. This involves projecting points onto constraints and iterating to equilibrium in a pseudo-dynamic fashion with damping. Based on this, a series of geometric constraints is defined to secure the interdependence between $\mathbf{F}^*$ and $\mathbf{F}$. Specifically, corresponding edges in the two diagrams ($\mathbf{f}_{ij}$ and $\mathbf{f}_{ij}^*$) are kept parallel to each other, while the vectors of the non-overlapping pairs of $\mathbf{F}^*$ are kept parallel and equal in magnitude. The time used by the solver to converge to an appropriate solution depends on the number of degrees of freedom of the nodes of the diagrams. To speed up the process, convenient nodes of $\mathbf{F}^*$ and $\mathbf{F}$ can be constrained and used as anchor points. In Figure 7, the transformation of the 3D force diagram $\mathbf{F}^*$ of a kinematically indeterminate network in equilibrium is shown, while the 3D form diagram $\mathbf{F}$ is modified accordingly. In particular, the node P_0^* of $\mathbf{F}^*$ is shifted using a translation vector $\mathbf{t}$ in order to increase the magnitudes of the forces connected to P_0^*.

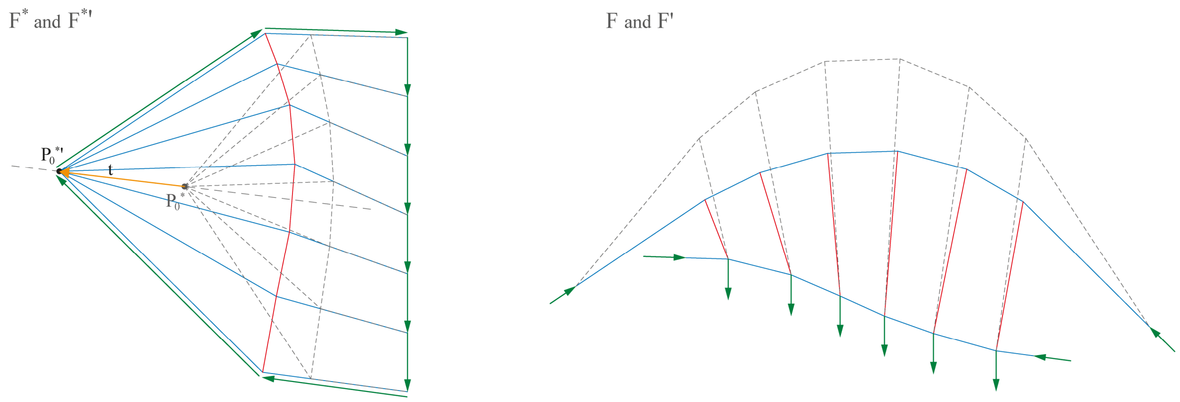

Figure 7: Local transformation of a 3D force diagram using numerical methods

5. Conclusions and Future Work

This paper has described a series of transformations applicable to the vector-based 3D force diagram of a given spatial network in equilibrium, such as a pin-jointed framework or a strut-and-tie model within a continuum of material. Global parallel, semi-global parallel and global projective transformations have been taken into consideration among those transformations that affect simultaneously all the elements of the force diagram or a subset of them. In relation to the transformations that modify individual elements of the diagram, three different approaches have been presented: the direct manipulation of kinematically determinate force diagrams, transformations using the Combinatorial Equilibrium Modelling (CEM) and constraint-driven transformations through numerical simulation.

As in the case of polyhedron-based 3D force diagrams (Akbarzadeh *et al.* [1] and Lee *et al.* [11]), the use of these transformations of vector-based 3D force diagrams opens up the possibility for an approach to structural design driven by the control of the forces. The manipulation of the forces within a spatial network, both in the case of global and local transformations, allows for a quick and interactive exploration of possible equilibrium solutions during the design process. Moreover, it facilitates the generation of creative structural configurations at an early design stage.

Further developments of the research will investigate the applicability of these transformations to actual structural design and analysis problems. In these cases, specific structural and boundary constraints will be taken into account and the opportunity to complement the transformations with suitable optimization processes will be evaluated. Moreover, further transformations of vector-based 3D force diagrams will be explored, including their hierarchical manipulation based on the combination and the subdivision of their force polygons.

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